Vacuum Energy Density Fluctuations in Minkowski and Casimir States via Smeared Quantum Fields and Point Separation

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Abstract

We present calculations of the variance of fluctuations and of the mean of the energy momentum tensor of a massless scalar field for the Minkowski and Casimir vacua as a function of an intrinsic scale defined by a smeared field or by point separation. We point out that contrary to prior claims, the ratio of variance to mean-squared being of the order unity is not necessarily a good criterion for measuring the invalidity of semiclassical gravity. For the Casimir topology we obtain expressions for the variance to mean-squared ratio as a function of the intrinsic scale (defined by a smeared field) compared to the extrinsic scale (defined by the separation of the plates, or the periodicity of space). Our results make it possible to identify the spatial extent where negative energy density prevails which could be useful for studying quantum field effects in worm holes and baby universe, and for examining the design fea-

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sibility of real-life ‘time-machines’. For the Minkowski vacuum we find that the ratio of the variance to the mean-squared, calculated from the coincidence limit, is identical to the value of the Casimir case at the same limit for spatial point separation while identical to the value of a hot flat space result with a temporal point-separation. We analyze the origin of divergences in the fluctuations of the energy density and discuss choices in formulating a procedure for their removal, thus raising new questions into the uniqueness and even the very meaning of regularization of the energy momentum tensor for quantum fields in curved or even flat spacetimes when spacetime is viewed as having an extended structure.

I. INTRODUCTION

Recent years saw the beginning of serious studies of the fluctuations of the energy momentum tensor (EMT) $\hat{T}_{\mu\nu}$ of quantum fields in spacetimes with boundaries [1] (such as Casimir effect [2]) [3,4], nontrivial topology (such as imaginary time thermal field theory) or nonzero curvature (such as the Einstein universe) [5]. A natural measure of the strength of fluctuations is χ [7], the ratio of the variance $\Delta\rho^2$ of fluctuations in the energy density (expectation value of the $\hat{\rho}^2$ operator minus the square of the mean $\hat{\rho}$ taken with respect to some quantum state) to its mean-squared (square of the expectation value of $\hat{\rho}$):

$$\chi \equiv \frac{\langle\hat{\rho}^2\rangle - \langle\hat{\rho}\rangle^2}{\langle\hat{\rho}\rangle^2} \equiv \frac{\Delta\rho^2}{\bar{\rho}^2} \tag{1.1}$$

Alternatively, we can use the quantity introduced by Kuo and Ford [4]

$$\Delta \equiv \frac{\langle\hat{\rho}^2\rangle - \langle\hat{\rho}\rangle^2}{\langle\hat{\rho}^2\rangle} = \frac{\chi}{\chi+1} \tag{1.2}$$

Assuming a positive definite variance $\Delta\rho^2 \geq 0$, then $0 \leq \chi \leq \infty$ and $0 \leq \Delta \leq 1$ always, with $\Delta \ll 1$ falling in the classical domain. Kuo and Ford (KF) displayed a number of quantum states (vacuum plus 2 particle state, squeezed vacuum and Casimir vacuum) with respect to which the expectation value of the energy momentum tensor (00 component) gives rise to negative local energy density. For these states Δ is of order unity. From this result they drew the implications, amongst other interesting inferences, that semiclassical gravity (SCG) [8] based on the semiclassical Einstein equation

$$G_{\mu\nu} = 8\pi G \left\langle \hat{T}_{\mu\nu} \right\rangle \tag{1.3}$$

(where $G_{\mu\nu}$ is the Einstein tensor and G the Newton gravitational constant) could become invalid under these conditions. The validity of semiclassical gravity in the face of fluctuations of quantum fields as source is an important issue which has caught the attention of many authors [9]. We hold a different viewpoint on this issue from KF, which we hope to clarify in this study.

In this series of papers, we would like to examine more closely this and related issues of SCG, such as the regularization in the energy momentum tensor of quantum fields and fluctuations of spacetime metric. In this paper we study a free field in flat space and spacetimes with boundaries. Later papers deal with curved spacetimes depicting the early universe (quantum fluctuations and structure formation) and black holes (horizon fluctuations and dynamical backreactions).

As explained elsewhere by one of us, when fluctuations of the energy momentum tensors are included as source for the dynamics of spacetime, these problems are best discussed in the larger context of stochastic semiclassical gravity (SSG) [10] program based on Einstein-Langevin type of equations [11], which is the proper framework to address Planck scale physics.

There are two groups of interrelated issues in quantum field theory in flat (ordinary QFT) or curved spacetimes (QFTCST), or semiclassical gravity (SCG –where the background spacetime dynamics is determined by the backreaction of the mean value of quantum fields): one pertaining to quantum fields and the other to spacetimes. We discuss the first set relating to the fluctuations of the EMT over its mean values with respect to the vacuum state. It strikes us as no great surprise that states which are more quantum (e.g., squeezed states) in nature than classical (e.g., coherent states) [12] may lead to large fluctuations comparable to the mean in the energy density. This can be seen even in the ratio of expectation values of moments of the displacement operators in simple quantum harmonic oscillators [13]. Such a condition exists peacefully with the underlying spacetime at least at the low energy of today's universe. We don't see sufficient ground to question the validity of SCG at energy below the Planck energy when the spacetime is depictable by a manifold structure, approximated locally by the Minkowski space. Besides, the cases studied in Kuo and Ford [4] as well as many others [5,9] are of a test-field nature, where backreaction is not considered. (So KF's criterion pertains more to QFTCST than to SCG, where in the former the central issue is compatibility, which is a weaker condition than consistency in the latter.)

To assess this situation we aim at calculating the variance of fluctuations to mean-squared ratio of a quantum field for the simplest case of Minkowski spacetime i.e., for ordinary quantum field theory. We find that $\Delta = 2/5$. This is a clear-cut counter-example to the claim of KF, since $\Delta = O(1)$ holds also for Minkowski space, where SCG is known to be valid at large scales. We view this situation as arising from the quantum nature of the vacuum state and saying little about the compatibility of the field source with the spacetime the quantum field lives in. In contrast, our view on this issue is that one should refer to a scale (of interaction or for probing accuracy) when measuring the validity of SCG. The conventional belief is that when reaching the Planck scale from below, QFTCST will break down because, amongst other things happening, graviton production at that energy will become significant so as to render the classical background spacetime unstable, and the mean value of quantum field taken as a source for the Einstein equation becomes inadequate. To address this issues as well as the issue of the spatial extent where negative energy density can exist, we view it necessary to introduce a scale in the spacetime regions where quantum fields are defined to monitor how the mean value and the fluctuations of the energy momentum tensor change. Point separation is an ideal method to adopt for this purpose.

In conventional field theories the stress tensor built from the product of a pair of field operators evaluated at a single point in the spacetime manifold is, strictly speaking, ill-defined. The point separation scheme [14] was introduced as a method of regularization of the energy momentum tensor for quantum fields in curved spacetime. In this scheme, one introduces an artificial separation of the single point x to a pair of closely separated points x and x'. The problematic terms involving field products such as $\hat{\phi}(x)^2$ becomes $\hat{\phi}(x)\hat{\phi}(x')$, whose expectation value is well defined. One then brings the two points back (taking the coincidence limit) to identify the divergences present, which will then be removed (regularization) or moved (by renormalizing the coupling constants), thereby obtaining a well-defined, finite stress tensor at a single point. In this context point separation was introduced as a trick for identifying the ultraviolet divergences in a covariant manner. One of us in the development of the stochastic semiclassical gravity program [10] has maintained the view (and as we will expound further in later papers) that instead of being used as a mere technical device in QFTCST, this method has much greater physical content. We prefer to view the operator valued EM bi-tensor $\hat{T}_{\mu\nu}(x, y)$ and EM two point function $\left\langle \hat{T}_{\mu\nu}(x, y) \right\rangle$ as the more fundamental objects in a more basic theory of spacetime and matter which has the point-defined quantum field theory as a low energy limit. This allows for spacetime to acquire an extended structure at sub-Planckian scale.[1]

For our stated purpose above, there is another way to introduce a scale in the quantum field theory, i.e., by introducing a (spatial) smearing function $f(\mathbf{x})$ to define smeared field operators $\hat{\phi}_t(f_{\mathbf{x}})$. In this paper we shall construct a scheme to encompass both aspects, by defining field operators at two separated points (connected by distance r) and using a Gaussian smearing function (with variance σ^2). We derive expressions for the EM bi-tensor

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[1]Viewing the two points here as the end points of an open string gives what one of us called a 'skeletal representation' of string theory, where the internal excitation modes are suppressed. Related to the so-called dipole approximation of string theory in recent literature [15] it should also carry features of non-commutative geometry of spacetime possible at the Planck scale.

operator, its mean and its fluctuations as functions of r, σ, for a massless scalar field in both the Minkowski and the Casimir spacetimes. The interesting result we find is that while both the vacuum expectation value and the fluctuations of energy density grow as $\sigma \to 0$, the ratio of the variance of the fluctuations to its mean-squared remains a constant χ_d (d is the spatial dimension of spacetime) which is independent of σ. The measure Δ_d $(= \chi_d/(\chi_d + 1))$ depends on the dimension of space and is of the order unity. It varies only slightly for spacetimes with boundary or nontrivial topology. For example Δ for Minkowski is 2/5, while for Casimir is 6/7 (cf, from [5]). Add to this our prior result for the Einstein Universe, $\Delta = 111/112$, independent of curvature, and that for hot flat space [16], we see a pattern emerging.

These results allow us to address two interrelated issues: 1) Fluctuations of the energy density and validity of semiclassical gravity, and 2) The spatial extent where negative energy density can exist. For Issue 1) we see that i) the fluctuations of the energy density as well as its mean both increase with decreasing distance (or probing scale), while ii) the ratio of the variance of the fluctuations in EMT to its mean-squared is of the order unity. We view the first but not the second feature as linked to the question of the validity of SCG –the case for Minkowski spacetime alone is sufficient to testify to the fallacy of Kuo and Ford's criterion. The second feature represents something quite different, pertaining more to the quantum nature of the vacuum state than to the validity of SCG.

For Issue 2) it is well known that negative energy density exists in Casimir geometry, moving mirrors, black holes and worm holes. Proposals have also been conjured to use the negative energy density for the design of time machines [6]. Our results (Figures 1, 2) provide an explicit scale dependence of the regularized vacuum energy density $\rho_{L,reg}$ and its fluctuations $\Delta_{L,reg}$, specifically σ/L, the ratio of the smearing length (field scale) to that of the Casimir length (geometry scale). For example, Fig. 2 shows that only for $\sigma/L < 0.24$ is $\rho_{L,reg} < 0$. Recall σ gives the spatial extent the field is probed or smeared. Ordinary pointwise quantum field theory which probes the field only at a point does not carry information about the spatial extent where negative energy density sustains. These results have direct implications on wormhole physics (and time machines, if one gets really serious about these fictions [6]). If L is the scale characterizing the size ('throat') of the wormhole where one thinks negative energy density might prevail, and designers of 'time machines' wish to exploit for 'time-travel' , our result provides a limit on the size of the probe (spaceship in the case of time-travel) in ratio to L where such conditions may exist. It could also provide a quantum field-theoretical bound on the probability of spontaneous creation of baby universes from quantum field energy fluctuations.

An equally weighty issue brought to light in this study is 3) the meaning of regularization in the face of EMT fluctuations. Since we have the point-separated expressions of the EMT and its fluctuations we can study how they change as a function of separation or smearing. In particular we can see how divergences arise at the coincidence limit. Whether certain cross terms containing divergences have physical meaning is a question raised by the recent studies of Wu and Ford [18]. We can use these calculations to examine these issues and ask the broader question of what exactly regularization entails, where divergences arise and how they are to be treated. The consideration of divergences in the fluctuations of EMT requires a more sophisticated rationale and reveals a deeper layer of issues pertaining to effective versus more fundamental theories. If we view ordinary quantum field theory defined at points as a low energy limit of a theory of spacetime involving extended structures (such as string theory), then these results would shed light on their meaning and inter-connections.

In this paper we will discuss three aspects of quantum field theory in curved spacetime in the light of fluctuations of quantum stress energy : 1) Fluctuation to mean ratio of vacuum energy density and the validity of SCG; 2) The point separated results of the mean and the fluctuations of the energy density for two states: the Minkowski case which has no scale present (massless field) and the Casimir case which has a scale present (the separation of the plates); 3) The circumstances when and how divergences appear and the meaning of regularization in point-defined field theories versus theories defined at separated points and/or smeard fields. A summary of the main points of 1) has been given in [7]. In this paper we give details of the calculations and discuss the regularization issue 3), while leaving the issue 2) of the spatial extent of negative energy density and its implications for quantum effects of worm holes, baby universes and time travel to a future investigation. In Sec. 2 we define the smeared field operators and their products defined at separated points. In Sec. 3 we construct from these the smeared energy density and its fluctuations, and calculate the ratio of the fluctuations to the mean for a flat space (Minkowski geometry). In Sec. 4 we analyze the case for a Casimir geometry of one periodic spatial dimension. In Sec. 5 we consider the point-separation calculation in Minkowski space, comparing the different results obtained from taking the coincidence limit from temporal versus spatial directions. Finally in Sec. 6 we summarize the major results and discuss the meaning of our finding in relation to the issues raised above.

II. SMEARED FIELD OPERATORS AT SEPARATED POINTS

Since the field operator in conventional point-defined quantum field theory is an operator-valued distribution, products of field operators at a point become problematic. This parallels the problem with defining the square of a delta function $\delta^2(x)$. Distributions are defined via their integral against a test function: they live in the space dual to the test function space. By going from the field operator $\hat{\phi}(x)$ to its integral against a test function, $\hat{\phi}(f) = \int \hat{\phi}\, f$, we can now readily consider products.

When we take the test functions to be spatial Gaussians, we are smearing the field operator over a finite spatial region. Physically we see smearing as representing the necessarily finite extent of an observer's probe, or the intrinsic limit of resolution in carrying out a measurement at a low energy (compared to Planck scale). In contrast to the ordinary point-defined quantum field theory, where ultraviolet divergences occur in the energy momentum tensor, smeared fields give no ultraviolet divergence. This is because smearing is equivalent to a regularization scheme which imparts an exponential suppression to the high momentum modes and restricts the contribution of the high frequency modes in the mode sum.

With this in mind, we start by defining the spatially smeared field operator

$$\hat{\phi}_t(f_{\mathbf{x}}) = \int \hat{\phi}(t, \mathbf{x}') f_{\mathbf{x}}(\mathbf{x}') d\mathbf{x}' \tag{2.1}$$

where $f_{\mathbf{x}}(\mathbf{x}')$ is a suitably smooth function. With this, the two point operator becomes

$$\left(\hat{\phi}_t(f_{\mathbf{x}})\right)^2 = \int\int \hat{\phi}(t, \mathbf{x}')\hat{\phi}(t, \mathbf{x}'') f_{\mathbf{x}}(\mathbf{x}') f_{\mathbf{x}}(\mathbf{x}'') d\mathbf{x}\, d\mathbf{x}' \tag{2.2}$$

which is now finite. In terms of the vacuum $|0\rangle$ ($\hat{a}_{\mathbf{k}}\,|0\rangle = 0$, for all $\mathbf{k}$) we have the usual mode expansion

$$\hat{\phi}\,(t_1, \mathbf{x}_1) = \int d\mu\,(\mathbf{k}_1) \left(\hat{a}_{\mathbf{k}_1}\, u_{\mathbf{k}_1}(t_1, \mathbf{x}_1) + \hat{a}^\dagger_{\mathbf{k}_1}\, u^*_{\mathbf{k}_1}(t_1, \mathbf{x}_1)\right) \tag{2.3}$$

with

$$u_{\mathbf{k}_1}\,(t_1, \mathbf{x}_1) = N_{k_1} e^{i\,(\mathbf{k}_1\cdot\mathbf{x}_1 - t_1\,\omega_1)}, \quad \omega_1 = |\mathbf{k}_1|\,, \tag{2.4}$$

where the integration measure $\int d\mu\,(\mathbf{k}_1)$ and the normalization constants $N_{\mathbf{k}_1}$ are given for a Minkowski and Casimir spaces by (2.10) and (4.1) respectively.

In this work, we use a Gaussian smearing function

$$f_{\mathbf{x}_0}(\mathbf{x}) = \left(\frac{1}{4\,\pi\,\sigma^2}\right)^{\frac{d}{2}} e^{-\left(\frac{\mathbf{x}_0 - \mathbf{x}_1}{2\sigma}\right)^2} \tag{2.5}$$

with the properties $\int f_{\mathbf{x}_0}\,(\mathbf{x}')\, d\mathbf{x}' = 1$, $\int \mathbf{x}'\, f_{\mathbf{x}_0}\,(\mathbf{x}')\, d\mathbf{x}' = \mathbf{x}_0$ and $\int |\mathbf{x}'|^2\, f_{\mathbf{x}_0}\,(\mathbf{x}')\, d\mathbf{x}' = 2d\sigma^2 + |\mathbf{x}_0|^2$. Using

$$\begin{aligned} \int u_{\mathbf{k}_1}(t, \mathbf{x})\, f_{\mathbf{x}_1}(\mathbf{x}) d\mathbf{x} &= N_{k_1} e^{-it\omega_1} \prod_{i=1}^{d} \left(\frac{1}{2\,\sqrt{\pi}\,\sigma} \int e^{+ik_{1i}x_i - \left(\frac{x_{1i} - x_i}{2\sigma}\right)^2} dx_i\right) \\ &= N_{k_1} e^{-i\,t\,w + i\,\mathbf{k}_1\cdot\mathbf{x}_1 - \sigma^2\,{k_1}^2} \end{aligned} \tag{2.6}$$

we get the smeared field operator

$$\hat{\phi}_{t_1}\,(f_{\mathbf{x}_1}) = \int d\mu\,(\mathbf{k}_1)\, N_{k_1} e^{-i\,\mathbf{k}_1\cdot\mathbf{x}_1 - \sigma^2\,{k_1}^2 - i\,t_1\,\omega_1} \left(e^{2\,i\,\mathbf{k}_1\cdot\mathbf{x}_1}\,\hat{a}_{\mathbf{k}_1} + e^{2\,i\,t_1\,\omega_1}\,\hat{a}^\dagger_{\mathbf{k}_1}\right) \tag{2.7}$$

and their product:

$$\begin{aligned} \hat{\phi}_{t_1}\,(f_{\mathbf{x}_1})\,\hat{\phi}_{t_2}\,(f_{\mathbf{x}_2}) \;=& \int d\mu\,(\mathbf{k}_1, \mathbf{k}_2)\, N_{k_1}\, N_{k_2} e^{-\sigma^2\left({k_1}^2 + {k_2}^2\right) - i\,(\mathbf{k}_1\cdot\mathbf{x}_1 + \mathbf{k}_2\cdot\mathbf{x}_2 + t_1\,\omega_1 + t_2\,\omega_2)} \\ & \left(e^{2\,i\,(\mathbf{k}_1\cdot\mathbf{x}_1 + \mathbf{k}_2\cdot\mathbf{x}_2)}\,\hat{a}_{\mathbf{k}_1}\hat{a}_{\mathbf{k}_2} + e^{2\,i\,(\mathbf{k}_1\cdot\mathbf{x}_1 + t_2\,\omega_2)}\,\hat{a}_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}_2} + e^{2\,i\,(\mathbf{k}_2\cdot\mathbf{x}_2 + t_1\,\omega_1)}\,\hat{a}^\dagger_{\mathbf{k}_1}\hat{a}_{\mathbf{k}_2}\right. \\ & \left. + e^{2\,i\,(t_1\,\omega_1 + t_2\,\omega_2)}\,\hat{a}^\dagger_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}_2}\right). \end{aligned} \tag{2.8}$$

The smearing has introduced a factor $e^{-\sigma^2 k^2}$ for each of the momenta. This factor acts as the regulator for the mode sums; in this sense the high momenta that led to the divergences have been controlled. We also note the $\sigma \to 0$ limit amounts to the relaxation of the regulator and the point field theory version of the field operator is recovered.

A. Coincident Smeared Green Function

We can let the two spatial points come together, $\mathbf{x}_2 = \mathbf{x}_1 = 0$, and get the coincident limit of the Green function

$$\begin{aligned} G\left(\Delta t\right) &= \left\langle 0 \left| \hat{\phi}_{t_1}\left(f_{\mathbf{x}_1}\right) \hat{\phi}_{t_2}\left(f_{\mathbf{x}_1}\right) \right| 0 \right\rangle \\ &= \int d\mu\left(\mathbf{k}_1\right) N_{k_1}^2 e^{-2\,\sigma^2\,k_1^2 + i\,\Delta t\,\omega_1} \end{aligned} \tag{2.9}$$

For flat space the normalization and integration measure are

$$N_{k_1}^2 = \frac{1}{2^{d+1}\pi^d\,\omega_1} \quad \text{and} \quad \int d\mu(\mathbf{k}_1) = \frac{2\,\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty k_1^{d-1} dk_1 \tag{2.10}$$

and the Green function is

$$\begin{aligned} G(\Delta t) = \frac{1}{2^{\frac{3}{2}\,(d+1)}\,\pi^{\frac{d}{2}}\,\sigma^d\,\Gamma\left(\frac{d}{2}\right)} \Bigg\{ & 2\,\sigma\,\Gamma\left(\frac{d-1}{2}\right)\, {}_1F_1\left(\frac{d-1}{2}; \frac{1}{2}; -\frac{\Delta t^2}{8\,\sigma^2}\right) \\ & + i\,\sqrt{2}\,\Delta t\,\Gamma\left(\frac{d}{2}\right)\, {}_1F_1\left(\frac{d}{2}; \frac{3}{2}; -\frac{\Delta t^2}{8\,\sigma^2}\right) \Bigg\} \end{aligned} \tag{2.11a}$$

$$= \frac{\Gamma\left(\frac{d-1}{2}\right)}{2^{\frac{3d+1}{2}}\,\pi^{\frac{d}{2}}\sigma^{d-1}\,\Gamma\left(\frac{d}{2}\right)} \left(1 + \frac{i\,\Delta t\,\Gamma\left(\frac{d}{2}\right)}{\sqrt{2}\,\sigma\,\Gamma\left(\frac{d-1}{2}\right)} - \frac{(d-1)\,\Delta t^2}{8\,\sigma^2}\right) + O\left(\Delta t^3\right), \tag{2.11b}$$

which we see is finite for this spatial coincident limit, and is finite for the $\Delta t \to 0$ limit as well. For spatial dimension $d = 3$, the smeared Green function is

$$G(\Delta t) = \frac{1}{16\,\pi^2\,\sigma^2} - \frac{\Delta t\,\left(\mathrm{Erfi}\left(\frac{\Delta t}{2\,\sqrt{2}\,\sigma}\right) - i\right)}{32\,\sqrt{2}\,e^{\frac{\Delta t^2}{8\,\sigma^2}}\,\pi^{\frac{3}{2}}\,\sigma^3} \tag{2.12a}$$

$$= \frac{1}{16\,\pi^2\,\sigma^2}\left(1 + \frac{i\Delta t\,\sqrt{\pi}}{2\sqrt{2}\sigma} - \frac{\Delta t^2}{4\,\sigma^2}\right) + O\left(\Delta t^3\right) \tag{2.12b}$$

B. Point-Separated Smeared Energy Density Operator

For a classical scalar function, the energy density is

$$\rho\left(t_1,\mathbf{x}_1\right)=\frac{1}{2}\left(\left(\partial_{t_1}\phi\right)^2+\left(\vec{\nabla}\phi\right)^2\right) \tag{2.13}$$

We cannot go directly to the quantum field case since the energy density has pairs of field operators evaluated at the same point. We can however define an energy density operator which contains smeared field operators at separated points. Then by taking the coincident ($\mathbf{x}_1 \to \mathbf{x}_2$) limit, we will be using the smearing to regularize the energy density, while if we take the zero smearing width ($\sigma \to 0$) limit, we are using point separation regularization.

Point separation consists of symmetrically splitting the operator product as

$$\hat{\phi}(t_1,\mathbf{x}_1)^2 \to \frac{1}{2}\left(\hat{\phi}(t_1,\mathbf{x}_1)\hat{\phi}(t_2,\mathbf{x}_2)+\hat{\phi}(t_2,\mathbf{x}_2)\hat{\phi}(t_1,\mathbf{x}_1)\right). \tag{2.14}$$

Products of derivatives of the field operator are symmetrically split according to, e.g., time derivatives become

$$\left(\partial_{t_1}\hat{\phi}(t_1,\mathbf{x}_1)\right)^2 \to \frac{1}{2}\left(\left(\partial_{t_1}\hat{\phi}(t_1,\mathbf{x}_1)\right)\left(\partial_{t_2}\hat{\phi}(t_2,\mathbf{x}_2)\right)+\left(\partial_{t_2}\hat{\phi}(t_2,\mathbf{x}_2)\right)\left(\partial_{t_1}\hat{\phi}(t_1,\mathbf{x}_1)\right)\right) \tag{2.15}$$

We introduce the smeared field operator derivatives

$$\begin{aligned}\left(\partial_{t_1}\hat{\phi}_{t_1}\right)\left(f_{\mathbf{x}_1}\right) &= \int\left(\partial_{t_1}\hat{\phi}\left(t_1,\mathbf{x}'\right)\right)f_{\mathbf{x}_1}\left(\mathbf{x}'\right)\,d\mathbf{x}' \\ &= i\int d\mu(\mathbf{k}_1)N_{k_1}\,\omega_1 e^{-i\,\mathbf{k}_1\cdot\mathbf{x}_1-\sigma^2\,{k_1}^2-i\,t_1\,\omega_1}\left(e^{2\,i\,t_1\,\omega_1}\,\hat{a}^\dagger_{\mathbf{k}_1}-e^{2\,i\,\mathbf{k}_1\cdot\mathbf{x}_1}\,\hat{a}_{\mathbf{k}_1}\right)\end{aligned} \tag{2.16a}$$

$$\begin{aligned}\left(\vec{\nabla}_{\mathbf{x}_1}\hat{\phi}_{t_1}\right)\left(f_{\mathbf{x}_1}\right) &= \int\left(\vec{\nabla}_{\mathbf{x}'}\hat{\phi}\left(t_1,\mathbf{x}'\right)\right)f_{\mathbf{x}_1}\left(\mathbf{x}'\right)\,d\mathbf{x}' \\ &= -i\int d\mu(\mathbf{k}_1)\mathbf{k}_1\,N_{k_1}e^{-i\,\mathbf{k}_1\cdot\mathbf{x}_1-\sigma^2\,{k_1}^2-i\,t_1\,\omega_1}\left(e^{2\,i\,t_1\,\omega_1}\,\hat{a}^\dagger_{\mathbf{k}_1}-e^{2\,i\,\mathbf{k}_1\cdot\mathbf{x}_1}\,\hat{a}_{\mathbf{k}_1}\right)\end{aligned} \tag{2.16b}$$

and use the symmetric splitting to define the point separated smeared energy density operator

$$\hat{\rho}\left(t_1,\mathbf{x}_1;t_2,\mathbf{x}_2;\sigma\right) = \frac{1}{4}\Big\{\left(\left(\partial_{t_1}\hat{\phi}_{t_1}\right)\left(f_{\mathbf{x}_1}\right)\right)\left(\left(\partial_{t_2}\hat{\phi}_{t_2}\right)\left(f_{\mathbf{x}_2}\right)\right)+\left(\left(\partial_{t_2}\hat{\phi}_{t_2}\right)\left(f_{\mathbf{x}_2}\right)\right)\left(\left(\partial_{t_1}\hat{\phi}_{t_1}\right)\left(f_{\mathbf{x}_1}\right)\right)$$

$$
+\left(\left(\vec{\nabla}_{\mathbf{x}_1}\hat{\phi}_{t_1}\right)(f_{\mathbf{x}_1})\right)\left(\left(\vec{\nabla}_{\mathbf{x}_2}\hat{\phi}_{t_2}\right)(f_{\mathbf{x}_2})\right)+\left(\left(\vec{\nabla}_{\mathbf{x}_2}\hat{\phi}_{t_2}\right)(f_{\mathbf{x}_2})\right)\left(\left(\vec{\nabla}_{\mathbf{x}_1}\hat{\phi}_{t_1}\right)(f_{\mathbf{x}_1})\right)\Big\}
$$

$$
\begin{aligned}
&= -\frac{1}{4}\int d\mu(\mathbf{k}_1)\, d\mu(\mathbf{k}_2) N_{k_1}\, N_{k_2}\, \left(\mathbf{k}_1\cdot\mathbf{k}_2+\omega_1\,\omega_2\right) \\
&\quad \times e^{-i(\mathbf{k}_1\cdot\mathbf{x}_1+\mathbf{k}_2\cdot\mathbf{x}_2)-i(t_1\,\omega_1+t_2\,\omega_2)-\sigma^2\left({k_1}^2+{k_2}^2\right)} \\
&\quad \times \Big(e^{2\,i\,\mathbf{k}_1\cdot\mathbf{x}_1+2\,i\,\mathbf{k}_2\cdot\mathbf{x}_2}\,\hat{a}_{\mathbf{k}_1}\hat{a}_{\mathbf{k}_2} - e^{2\,i\,\mathbf{k}_1\cdot\mathbf{x}_1+2\,i\,t_2\,\omega_2}\,\hat{a}_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}_2} \\
&\qquad + e^{2\,i\,\mathbf{k}_1\cdot\mathbf{x}_1+2\,i\,\mathbf{k}_2\cdot\mathbf{x}_2}\,\hat{a}_{\mathbf{k}_2}\hat{a}_{\mathbf{k}_1} - e^{2\,i\,\mathbf{k}_2\cdot\mathbf{x}_2+2\,i\,t_1\,\omega_1}\,\hat{a}_{\mathbf{k}_2}\hat{a}^\dagger_{\mathbf{k}_1} \\
&\qquad - e^{2\,i\,\mathbf{k}_2\cdot\mathbf{x}_2+2\,i\,t_1\,\omega_1}\,\hat{a}^\dagger_{\mathbf{k}_1}\hat{a}_{\mathbf{k}_2} + e^{2\,i\,t_1\,\omega_1+2\,i\,t_2\,\omega_2}\,\hat{a}^\dagger_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}_2} \\
&\qquad - e^{2\,i\,\mathbf{k}_1\cdot\mathbf{x}_1+2\,i\,t_2\,\omega_2}\,\hat{a}^\dagger_{\mathbf{k}_2}\hat{a}_{\mathbf{k}_1} + e^{2\,i\,t_1\,\omega_1+2\,i\,t_2\,\omega_2}\,\hat{a}^\dagger_{\mathbf{k}_2}\hat{a}^\dagger_{\mathbf{k}_1}\Big)
\end{aligned} \tag{2.17}
$$

Its vacuum expectation value is

$$
\begin{aligned}
\rho\left(t_1,\mathbf{x}_1;t_2,\mathbf{x}_2;\sigma\right) &= \left\langle 0\left|\hat{\rho}_t\left(t_1,\mathbf{x}_1;t_2,\mathbf{x}_2;\sigma\right)\right|0\right\rangle \\
&= \frac{1}{2}\int d\mu\left(\mathbf{k}_1\right) N_{k_1}^2\,\omega_1^2\, e^{-2\,\sigma^2\,k_1^2} \\
&\qquad \times\left(e^{-i\left(\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{x}_2)-(t_1-t_2)\,\omega_1\right)}+e^{i\left(\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{x}_2)-(t_1-t_2)\,\omega_1\right)}\right)
\end{aligned} \tag{2.18}
$$

We obtain the smeared vacuum energy density by taking the coincidence

$$
\rho\left(\sigma\right) = \int d\mu\left(\mathbf{k}_1\right) N_{k_1}^2\,\omega_1^2\, e^{-2\,\sigma^2\,k_1^2} \tag{2.19}
$$

while the point separation expression is obtained from taking the zero smearing-width limit:

$$
\rho\left(t_1,\mathbf{x}_1;t_2,\mathbf{x}_2\right) = \int d\mu(\mathbf{k}_1) N_{k_1}^2\,\omega_1^2\,\cos(\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{x}_2)-(t_1-t_2)\,\omega_1) \tag{2.20}
$$

C. Point-Separated Smeared Energy Density Correlation Function

We now consider the point separated vacuum correlation function for the energy density operator:

$$
\begin{aligned}
\Delta\rho^2(t_1,\mathbf{x}_1,t_1',\mathbf{x}_1';t_2,\mathbf{x}_2,t_2',\mathbf{x}_2';\sigma) &= \left\langle 0\left|\hat{\rho}\left(t_1,\mathbf{x}_1;t_1',\mathbf{x}_1';\sigma\right)\hat{\rho}\left(t_2,\mathbf{x}_2;t_2',\mathbf{x}_2';\sigma\right)\right|0\right\rangle \\
&\quad -\rho\left(t_1,\mathbf{x}_1;t_1',\mathbf{x}_1';\sigma\right)\rho\left(t_2,\mathbf{x}_2;t_2',\mathbf{x}_2';\sigma\right)
\end{aligned} \tag{2.21}
$$

With this definition, the vacuum energy density correlation function is

$$\Delta\rho^2(t_1,\mathbf{x}_1;t_2,\mathbf{x}_2) \equiv \langle 0\,|\hat\rho\,(t_1,\mathbf{x}_1)\,\hat\rho\,(t_2,\mathbf{x}_2)|\,0\rangle - \langle 0\,|\hat\rho\,(t_1,\mathbf{x}_1)|\,0\rangle\,\langle 0\,|\hat\rho\,(t_2,\mathbf{x}_2)|\,0\rangle$$
$$= \Delta\rho^2\,(t_1,\mathbf{x}_1,t_1,\mathbf{x}_1;t_2,\mathbf{x}_2,t_2,\mathbf{x}_2;\sigma=0) \tag{2.22}$$

Since the divergences present in $\langle 0\,|\hat\rho\,(t_1,\mathbf{x}_1)\,\hat\rho\,(t_2,\mathbf{x}_2)|\,0\rangle$ for $(t_2,\mathbf{x}_2)\neq(t_1,\mathbf{x}_1)$ are canceled by those due to $\langle 0\,|\hat\rho\,(t_1,\mathbf{x}_1)|\,0\rangle$ and $\langle 0\,|\hat\rho\,(t_2,\mathbf{x}_2)|\,0\rangle$, we can assume $(t_1',\mathbf{x}_1')=(t_1,\mathbf{x}_1)$ and $(t_2',\mathbf{x}_2')=(t_2,\mathbf{x}_2)$ from the start. This will be confirmed during the computation of the vacuum expectation value.

First we consider just the square of the energy density operator; its expectation value is

$$\begin{aligned}\left\langle 0\left|\hat\rho^2\right|0\right\rangle = \frac{1}{4}\int d\mu\,(\mathbf{k}_1,\mathbf{k}_2)\,N_{k_1}^2\,N_{k_2}^2\,e^{-2\,\sigma^2\,\left(k_1^2+k_2^2\right)}\,\{&\\
(\mathbf{k}_1\cdot\mathbf{k}_2+\omega_1\,\omega_2)^2\Big(e^{i\left(\mathbf{k}_1\cdot\left(\mathbf{x}_1-\mathbf{x}_2'\right)+\mathbf{k}_2\cdot\left(\mathbf{x}_1'-\mathbf{x}_2\right)\right)-i\left(\left(t_1-t_2'\right)\omega_1+\left(t_1'-t_2\right)\omega_2\right)}&\\
+e^{i\left(\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{x}_2)+\mathbf{k}_2\cdot\left(\mathbf{x}_1'-\mathbf{x}_2'\right)\right)-i\left((t_1-t_2)\,\omega_1+\left(t_1'-t_2'\right)\omega_2\right)}\Big)&\\
+\Big[\omega_1^2\Big(e^{-i\left(\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{x}_2)-\left(t_1-t_1'\right)\omega_1\right)}+e^{i\left(\mathbf{k}_1\cdot\left(\mathbf{x}_1-\mathbf{x}_1'\right)-\left(t_1-t_1'\right)\omega_1\right)}\Big)&\\
\times\,\omega_2^2\Big(e^{-i\left(\mathbf{k}_2\cdot(\mathbf{x}_2-\mathbf{x}_4)-\left(t_2-t_2'\right)\omega_2\right)}+e^{i\left(\mathbf{k}_2\cdot\left(\mathbf{x}_2-\mathbf{x}_2'\right)-\left(t_2-t_2'\right)\omega_2\right)}\Big)\Big]\Big\}&\end{aligned} \tag{2.23}$$

By comparing the last two lines of the above expression with Eq.(2.18), we see this is but $\rho\,(t_1,\mathbf{x}_1;t_1',\mathbf{x}_1';\sigma)\,\rho\,(t_2,\mathbf{x}_2;t_2',\mathbf{x}_2';\sigma)$. Thus, the remainder is the desired expression for $\Delta\rho^2(t_1,\mathbf{x}_1,t_1',\mathbf{x}_1';t_2,\mathbf{x}_2,t_2',\mathbf{x}_2';\sigma)$. Even for the case $\sigma\to0$, this expression is finite for $(t_1',\mathbf{x}_1')\to(t_1,\mathbf{x}_1)$ and $(t_2',\mathbf{x}_2')\to(t_2,\mathbf{x}_2)$, as long as $(t_1,\mathbf{x}_1)\neq(t_2,\mathbf{x}_2)$. Letting $(t,\mathbf{x})=(t_2,\mathbf{x}_2)-(t_1,\mathbf{x}_1)$, our results for the energy density [from (2.18)] and its correlation function [from here] are

$$\rho\,(t,\mathbf{x};\sigma)=\int d\mu\,(\mathbf{k})\,N_k^2\,\omega^2\,e^{-2\,\sigma^2\,k^2}\,\cos(\mathbf{x}\cdot\mathbf{k}-t\,\omega) \tag{2.24}$$

$$\Delta\rho^2\,(t,\mathbf{x};\sigma)=\frac{1}{2}\int d\mu\,(\mathbf{k}_1,\mathbf{k}_2)\,N_{k_1}^2\,N_{k_2}^2\,(\mathbf{k}_1\cdot\mathbf{k}_2+\omega_1\,\omega_2)^2\,e^{-2\,\sigma^2\,\left(k_1^2+k_2^2\right)-i\,\mathbf{x}\cdot(\mathbf{k}_1+\mathbf{k}_2)+i\,t\,(\omega_1+\omega_2)} \tag{2.25}$$

III. SMEARED-FIELD ENERGY DENSITY AND FLUCTUATIONS IN MINKOWSKI SPACE

We consider a Minkowski space $R^1 \times R^d$ with d-spatial dimensions. For this space the mode density is

$$\int d\mu(\mathbf{k}) = \int_0^\infty k^{d-1}\, dk \int_{S^{d-1}} d\Omega_{d-1} \quad \text{with} \quad \int_{S^{d-1}} d\Omega_{d-1} = \frac{2\,\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \tag{3.1}$$

and the mode function normalization constant is $N_{k_1} = 1/\sqrt{2^{d+1}\,\pi^d\,\omega_1}$. We introduce the angle between two momenta in phase space, γ, via

$$\mathbf{k}_1 \cdot \mathbf{k}_2 = k_1 k_2 \cos(\gamma) = \omega_1\omega_2 \cos(\gamma). \tag{3.2}$$

The averages of the cosine and cosine squared of this angle over a pair of unit spheres are

$$\int_{S^{d-1}} d\Omega_1 \int_{S^{d-1}} d\Omega_2 \cos(\gamma) = 0 \tag{3.3a}$$

$$\int_{S^{d-1}} d\Omega_1 \int_{S^{d-1}} d\Omega_2 \cos^2(\gamma) = \frac{4\,\pi^d}{d\,\Gamma\left(\frac{d}{2}\right)^2}. \tag{3.3b}$$

The smeared energy density (2.19) becomes

$$\begin{aligned}\rho(\sigma) &= \frac{1}{2^d\,\pi^{\frac{d}{2}}\,\Gamma\left(\frac{d}{2}\right)} \int_0^\infty \frac{{k_1}^d}{e^{2\,\sigma^2\,{k_1}^2}}\, dk_1 \\ &= \frac{\Gamma\left(\frac{d+1}{2}\right)}{2^{\frac{3\,(d+1)}{2}}\,\pi^{\frac{d}{2}}\,\sigma^{d+1}\Gamma\left(\frac{d}{2}\right)}\end{aligned} \tag{3.4}$$

For the fluctuations of the smeared energy density operator, we evaluate (2.25) for this space and find

$$\begin{aligned}\Delta\rho^2(\sigma) &= \frac{1}{2^{(2d+3)}\pi^{2\,d}} \int_0^\infty \int_0^\infty \int_{S^{d-1}} \int_{S^{d-1}} \frac{\left(1+\cos(\gamma)\right)^2 k_1^d\, k_2^d}{e^{2\,\sigma^2\,\left(k_1^2+k_2^2\right)}}\, d\Omega_1\, d\Omega_2\, dk_1\, dk_2 \\ &= \frac{(d+1)\,\Gamma\left(\frac{d+1}{2}\right)^2}{2^{(3d+4)}\,d\,\pi^d\,\sigma^{2\,(d+1)}\,\Gamma\left(\frac{d}{2}\right)^2}\end{aligned} \tag{3.5}$$

Defining the (dimension-dependent) constant

$$\chi_d \equiv \frac{1+d}{2\,d}, \tag{3.6}$$

we write the smeared fluctuations in terms of the square of the smeared energy density

$$\Delta\rho^2(\sigma) = \chi_d \, \rho(\sigma)^2 \tag{3.7}$$

We introduce the dimensionless measure of fluctuations

$$\Delta = \left| 1 - \frac{\langle\rho\rangle^2}{\langle\rho^2\rangle} \right| = \left| \frac{\Delta\rho^2}{\Delta\rho^2 + \langle\rho\rangle^2} \right| = \frac{\chi_d}{\chi_d + 1} \tag{3.8}$$

and for Minkowski space we have

$$\Delta_{\text{Minkowski}}(d) = \frac{1+d}{1+3\,d} \tag{3.9}$$

which has the particular values

d	1	3	5	∞
$\Delta_{\text{Minkowski}}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{3}{8}$	$\frac{1}{3}$

IV. SMEARED-FIELD IN CASIMIR TOPOLOGY

The Casimir topology is obtained from a flat space (with d spatial dimensions, i.e., $R^1 \times R^d$) by imposing periodicity L in one of its spatial dimensions, say, z, thus endowing it with a $R^1 \times R^{d-1} \times S^1$ topology. We decompose $\mathbf{k}$ into a component along the periodic dimension and call the remaining components $\mathbf{k}_\perp$:

$$\mathbf{k} = \left(\mathbf{k}_\perp, \frac{2\pi n}{L}\right) = (\mathbf{k}_\perp, ln)\,, l \equiv 2\pi/L \tag{4.1a}$$

$$\omega_1 = \sqrt{k_1^2 + l^2\, {n_1}^2} \tag{4.1b}$$

The normalization and momentum measure are

$$\int d\mu(\mathbf{k}) = \int_0^\infty k^{d-2}\, dk \int_{S^{d-2}} d\Omega_{d-2} \sum_{n=-\infty}^{\infty} \tag{4.1c}$$

$$N_{k_1} = \frac{1}{\sqrt{2^d\, L\, \pi^{d-1}\, \omega_1}} \tag{4.1d}$$

With this, the energy density (2.19) becomes

$$
\begin{aligned}
\rho_L(\sigma) &= \frac{l}{2^d\,\pi^{\frac{d+1}{2}}\,\Gamma\!\left(\frac{d-1}{2}\right)}\sum_{n_1=-\infty}^{\infty}\int_0^\infty k_1^{d-2}\,\left(k_1^2+l^2\,n_1^2\right)^{\frac{1}{2}}\,e^{-2\,\sigma^2\,\left(k_1^2+l^2\,n_1^2\right)}\,dk_1\\
&= \frac{l}{2^d\,\pi^{\frac{d+1}{2}}\,\Gamma\!\left(\frac{d-1}{2}\right)}\sum_{n_1=-\infty}^{\infty}\int_0^\infty \frac{{k_1}^d}{\sqrt{k_1^2+l^2\,n_1^2}}e^{-2\,\sigma^2\,\left(k_1^2+l^2\,n_1^2\right)}\,dk_1\\
&\quad+\frac{l}{2^d\,\pi^{\frac{d+1}{2}}\,\Gamma\!\left(\frac{d-1}{2}\right)}\sum_{n_1=-\infty}^{\infty}\int_0^\infty \frac{l^2\,{k_1}^{-2+d}\,n_1^2}{\sqrt{k_1^2+l^2\,n_1^2}}e^{-2\,\sigma^2\,\left(k_1^2+l^2\,n_1^2\right)}\,dk_1
\end{aligned}
\tag{4.2}
$$

Using the analysis of the Appendix A we write this as the sum of the two smeared Green function derivatives

$$
\begin{aligned}
\rho_L(\sigma) &= \left\langle 0_L\left|\left(\left(\nabla_\perp\phi_t\right)\left(f_{\mathbf{x}}\right)\right)^2\right|0_L\right\rangle+\left\langle 0_L\left|\left(\left(\partial_z\phi_t\right)\left(f_{\mathbf{x}}\right)\right)^2\right|0_L\right\rangle\\
&= G_L(\sigma)_{,x_\perp x_\perp}+G_L(\sigma)_{,zz}
\end{aligned}
\tag{4.3}
$$

where $|\,0_L\rangle$ is the Casimir vacuum.

A. Regularized Casimir Energy Density

Since $G_L(\sigma)_{,i}=G^{\mathrm{div}}_{L,i}+G^{\mathrm{fin}}_{L,i}$ ($i=x_\perp x_\perp$ or zz) we see how to split the smeared energy density into a $\sigma\to 0$ divergent term and the finite contribution:

$$
\rho_L(\sigma)=\rho_L^{\mathrm{div}}+\rho_L^{\mathrm{fin}}
\tag{4.4}
$$

where

$$
\begin{aligned}
\rho_L^{\mathrm{div}} &= G^{\mathrm{div}}_{L,x_\perp x_\perp}+G^{\mathrm{div}}_{L,zz}\\
&= \frac{\Gamma\!\left(\frac{d+1}{2}\right)}{2^{\frac{3\,(d+1)}{2}}\,\pi^{\frac{d}{2}}\,\sigma^{d+1}\Gamma\!\left(\frac{d}{2}\right)}\\
&= \rho(\sigma)
\end{aligned}
\tag{4.5}
$$

and

$$
\begin{aligned}
\rho_L^{\mathrm{fin}} &= G^{\mathrm{fin}}_{L,x_\perp x_\perp}+G^{\mathrm{fin}}_{L,zz}\\
&= -\frac{d\,\Gamma\!\left(-\frac{d}{2}\right)\Gamma\!\left(\frac{d}{2}\right)}{(4\pi)^{(d+3)/2}\,l^{d+1}}\sum_{p=1}^{\infty}(-1)^p\,(2\,l)^{2p}\,p\,(2p-1)^2\,\sigma^{2\,(p-1)}\frac{B_{2p+d-1}}{2p+d-1}\frac{(2p-3)!!}{(2\,p)!}\frac{\Gamma\!\left(p-\frac{1}{2}\right)}{\Gamma\!\left(p+\frac{d}{2}\right)}
\end{aligned}
\tag{4.6}
$$

With this we define the regularized energy density

$$\rho_{L,\mathrm{reg}} \equiv \lim_{\sigma\to 0} \left(\rho_L(\sigma) - \rho(\sigma)\right) = \frac{d\,\pi^{\frac{d}{2}}\,B_{d+1}\,\Gamma\!\left(-\frac{d}{2}\right)\,\Gamma\!\left(\frac{d}{2}\right)}{2\,(d+1)\,L^{d+1}\,\Gamma\!\left(\frac{d}{2}+1\right)} \tag{4.7}$$

and get the usual results

d	1	3	5
$\rho_{L,\mathrm{reg}}$	$-\frac{\pi}{6\,L^2}$	$-\frac{\pi^2}{90\,L^4}$	$-\frac{2\,\pi^3}{945\,L^6}$

B. Casimir energy density fluctuations

For the d-dimensional Casimir geometry, the fluctuations are

$$\begin{aligned}\Delta\rho_L^2(\sigma) &= \frac{l^2}{2^{2d+3}\,\pi^{2\,d}} \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} \int_0^\infty k_1^{d-2}\,dk_1 \int_0^\infty k_2^{d-2}\,dk_2 \int_{S^{d-2}} d\Omega_1 \int_{S^{d-2}} d\Omega_2 \\ &\quad\times \frac{e^{-2\,\sigma^2\left(\omega_1^2+\omega_2^2\right)}}{\omega_1\,\omega_2} \left(\cos(\gamma)\,k_1\,k_2 + l^2\,n_1\,n_2 + \omega_1\,\omega_2\right)^2 \\ &= \frac{l^2}{2^{2\,d}\,\pi^{d+1}\,\Gamma\!\left(\frac{d-1}{2}\right)^2} \sum_{n_1,n_2=-\infty}^{\infty} \int_0^\infty\!\!\int_0^\infty dk_1\,dk_2\, \frac{k_1^{d+2} e^{-2\,\sigma^2\,\left(k_1^2+l^2\,n_1^2\right)}}{\sqrt{k_1^2+l^2\,n_1^2}} \frac{k_2^{d+2} e^{-2\,\sigma^2\,\left(k_2^2+l^2\,n_2^2\right)}}{\sqrt{k_2^2+l^2\,n_2^2}} \\ &\quad\times \left(\frac{d\,k_1^2\,k_2^2}{2\,(d-1)} + \frac{l^2}{2}\left(k_2^2\,n_1^2 + k_1^2\,n_2^2\right) + l^4\,n_1^2\,n_2^2\right)\end{aligned} \tag{4.8}$$

We write this expression in terms of products of the Green functions derivatives used above:

$$\Delta\rho_L^2(\sigma) = \frac{d\,\left(G_L(\sigma)_{,x_\perp x_\perp}\right)^2}{2\,(d-1)} + G_L(\sigma)_{,zz}\left(G_L(\sigma)_{,x_\perp x_\perp} + G_L(\sigma)_{,zz}\right) \tag{4.9}$$

We can split $\Delta\rho_L^2(\sigma)$ into three general terms

$$\Delta\rho_L^2(\sigma) = \Delta\rho_L^{2,\mathrm{div}} + \Delta\rho_L^{2,\mathrm{cross}} + \Delta\rho_L^{2,\mathrm{fin}} \tag{4.10}$$

The first term contains only the divergent parts of the Green functions while the last term contains only the finite parts. This is similar to the split we used for the smeared energy density above. What is new here is the middle term $\Delta\rho_L^{2,\mathrm{cross}}$. This comes about from the products of the divergent part of one Green function and the finite part of the other. That this term arises for computations of the energy density fluctuations is a generic feature. We will discuss in greater detail the meaning of this term later.

The results of Appendix A give

$$\begin{aligned}
\Delta\rho_L^{2,\mathrm{div}} &= \frac{d\left(G^{\mathrm{div}}_{L,x_\perp x_\perp}\right)^2}{2\,(d-1)} + G^{\mathrm{div}}_{L,zz}\left(G^{\mathrm{div}}_{L,x_\perp x_\perp} + H_2^{\mathrm{div}}\right) \\
&= \frac{(d+1)\,\Gamma\left(\frac{d+1}{2}\right)^2}{d\,2^{3\,d+4}\,\pi^d\,\sigma^{2\,(d+1)}\,\Gamma\left(\frac{d}{2}\right)^2} \\
&= \chi_d\left(\rho_L^{\mathrm{div}}\right)^2 = \chi_d\left(\rho\left(\sigma\right)\right)^2
\end{aligned} \tag{4.11a}$$

$$\begin{aligned}
\Delta\rho_L^{2,\mathrm{cross}} &= \frac{d}{d-1}G^{\mathrm{div}}_{L,x_\perp x_\perp}G^{\mathrm{fin}}_{L,x_\perp x_\perp} + \left(2G^{\mathrm{div}}_{L,zz}G^{\mathrm{fin}}_{L,zz} + G^{\mathrm{div}}_{L,x_\perp x_\perp}G^{\mathrm{fin}}_{L,zz} + G^{\mathrm{div}}_{L,zz}G^{\mathrm{fin}}_{L,x_\perp x_\perp}\right) \\
&= \frac{(d+1)\,\Gamma\left(\frac{d+1}{2}\right)}{2^{\frac{3\,(d+1)}{2}}\,d\,\pi^{\frac{d}{2}}\,\sigma^{d+1}\,\Gamma\left(\frac{d}{2}\right)}\left(G^{\mathrm{fin}}_{L,x_\perp x_\perp} + G^{\mathrm{fin}}_{L,zz}\right) \\
&= 2\chi_d\,\rho_L^{\mathrm{div}}\,\rho_L^{\mathrm{fin}}
\end{aligned} \tag{4.11b}$$

$$\begin{aligned}
\Delta\rho_L^{2,\mathrm{fin}} &= \frac{d\left(G^{\mathrm{fin}}_{L,x_\perp x_\perp}\right)^2}{2\,(d-1)} + G^{\mathrm{fin}}_{L,zz}\left(G^{\mathrm{fin}}_{L,x_\perp x_\perp} + G^{\mathrm{fin}}_{L,zz}\right) \\
&= \frac{d^2\,l^{2(d+1)}}{2^{2d+7}\,\pi^{d+3}}\Gamma\left(-\frac{d}{2}\right)^2\Gamma\left(\frac{d}{2}\right)^2 \\
&\quad\times\sum_{p,q=1}^{\infty}\Bigg((-1)^{p+q}\,2^{2\,(p+q)}\,l^{2\,(p+q)}\,p\,q\,(2\,p-1)\,(2\,q-1)\,\sigma^{2\,(p+q-2)} \\
&\qquad\times\left(4+d^2-6\,q+d\,(2\,p+2\,q-3)+2\,p\,(4\,q-3)\right) \\
&\qquad\times\frac{B_{2\,p+d-1}\,B_{2\,q+d-1}\,(2\,p-3)!!\,(2\,q-3)!!}{(2\,p+d-1)\,(2\,q+d-1)\,(2\,p)!\,(2\,q)!}\frac{\Gamma\left(p-\frac{1}{2}\right)\,\Gamma\left(q-\frac{1}{2}\right)}{\Gamma\left(p+\frac{d}{2}\right)\,\Gamma\left(q+\frac{d}{2}\right)}\Bigg) \\
&\xrightarrow{\sigma\to 0} \frac{d^3\,\pi^d\,B_{d+1}^2\,\Gamma\left(-\frac{d}{2}\right)^2\Gamma\left(\frac{d}{2}\right)^2}{8\,(d+1)\,L^{2\,(d+1)}\,\Gamma\left(1+\frac{d}{2}\right)^2} \\
&= \frac{d\,(d+1)}{2}\left(\rho_{L,\mathrm{reg}}\right)^2
\end{aligned} \tag{4.11c}$$

From this we see the divergent and cross terms can be related to the smeared energy density via

$$\Delta\rho_L^{2,\mathrm{div}} + \Delta\rho_L^{2,\mathrm{cross}} = \chi_d\left\{\left(\rho_L^{\mathrm{div}}\right)^2 + 2\rho_L^{\mathrm{div}}\rho_L^{\mathrm{fin}}\right\} \tag{4.12}$$

where χ_d is the function that relates the fluctuations of the energy density to the mean energy density when the boundaries are not present, i.e., Minkowski space. This leads us to interpret these terms as due the vacuum fluctuations that are always present. With this in mind, we define the regularized fluctuations of the energy density

$$\begin{aligned}\Delta\rho^2_{L,\mathrm{reg}} &= \lim_{\sigma\to 0}\left(\Delta\rho^2_L(f) - \chi_d\left\{\left(\rho^{\mathrm{div}}_L\right)^2 + 2\rho^{\mathrm{div}}_L\rho^{\mathrm{fin}}_L\right\}\right)\\ &= \chi_{d,L}\left(\rho_{L,\mathrm{reg}}\right)^2 \end{aligned} \tag{4.13}$$

where

$$\chi_{d,L} \equiv \frac{d\,(d+1)}{2}. \tag{4.14}$$

We also define a regularized version of the dimensionless measure Δ:

$$\Delta_{L,\mathrm{Reg}} \equiv \frac{\Delta\rho^2_{L,\mathrm{Reg}}}{\Delta\rho^2_{L,\mathrm{Reg}} + \left(\rho_{L,\mathrm{Reg}}\right)^2} = \frac{d\,(d+1)}{2+d+d^2} \tag{4.15}$$

and note the values:

d	1	3	5	∞
$\Delta_{L,\mathrm{Reg}}$	$\frac{1}{2}$	$\frac{6}{7}$	$\frac{15}{16}$	1

Following the procedures described in Appendix B, we have made two plots, Fig. 1 of $\Delta(\sigma, L)$ and $\Delta_{L,Reg}$ versus σ/L, (which we call σ' here for short); and Fig. 2 of $\rho_{L,Reg}$ and $\sqrt{\Delta\rho^2_{L,Reg}}$ versus σ'. The range of σ' is limited to ≤ 0.4 because going any further would make the meaning of a local energy density ill-defined, as the smearing of the field extends to the Casimir boundary in space. (We believe this infrared limit also carry important physical meaning in reference to the structure of spacetime, it is outside the focus of this paper.)

Let us ponder on the meaning they convey. In Fig. 1, we first note that both curves are of the order unity. But the behavior of Δ (recall that the energy density fluctuations thus defined include the cross term along with the finite part and the state independent divergent part) is relatively insensitive to the smearing width, whereas $\Delta_{L,Reg}$, which measures only the finite part of the energy density fluctuations to the mean has more structure. In particular, it saturates its upper bound of 1 around $\sigma' = 0.24$. Note that if one adheres to the KF criterion [4] one would say that semiclassical gravity fails, but all that is happening here is that $\rho_{L,\text{Reg}} = 0$ while $\Delta\rho^2_{L,\text{Reg}}$ shows no special feature. The real difference between these two functions is the cross term, which is responsible for their markedly different structure and behavior. We have more to say about what to make of the cross term in the last section, which should be contrasted with the opinion of Wu and Ford [18] on its physical significance. In Fig. 2, the main feature to notice is that the regularized energy density crosses from negative to positive values at around $\sigma' = 0.24$. The negative Casimir energy density calculated in a point-wise field theory which corresponds to small ranges of σ' is expected, and is usually taken to signify the quantum nature of the Casimir state. As σ' increases we are averaging the field operator over a larger region, and thus sampling the field theory from the ultraviolet all the way to the infrared region. At large σ' finite size effect begins to set in. The difference and relation of these two effects are explained in [19]: Casimir effect arises from summing up the quantum fluctuations of ALL modes (as altered by the boundary), with no insignificant short wavelength contributions, whereas finite size effect has dominant contributions from the LONGEST wavelength modes, and thus reflect the large scale behavior. As the smearing moves from a small scale to the far boundary of space, the behavior of the system is expected to shift from a Casimir-dominated to a finite size-dominated effect. This could be the underlying reason in the crossover behavior of $\rho_{L,Reg}$.

V. POINT SEPARATED ENERGY DENSITY AND FLUCTUATIONS IN MINKOWSKI SPACE

We return now to the Minkowski space and consider the point-separated expressions for the energy density and its fluctuations. This will lead to some interesting new observations about point-separation and maybe even the extended structure of spacetime. Consider the point separated energy density (2.24) with $\sigma = 0$ in a Minkowski space with d-spatial dimensions

$$\rho(t,\mathbf{x}) = \int d\mu(\mathbf{k})\, N_k^2\, \omega^2\, \cos(\mathbf{x}\cdot\mathbf{k} - t\,\omega)$$
$$= \frac{1}{2^d\,\pi^{\frac{d+1}{2}}\Gamma\left(\frac{d-1}{2}\right)} \int_0^\infty \int_{-\infty}^\infty \cos\left(x\,k_x - t\,\sqrt{k_\perp^2 + k_x^2}\right)\, k_\perp^{d-2}\, \sqrt{k_\perp^2 + k_x^2}\, dk_x\, dk_\perp \tag{5.1}$$

where we take $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2 = x\hat{x}$ and decompose $\mathbf{k} = (k_x, \mathbf{k}_\perp)$ into one component along $\hat{x}$ and two perpendicular to $\hat{x}$. We change variables to $k_x = k\cos\phi$ and $k_\perp = |\mathbf{k}_\perp| = k\sin\phi$ and evaluate

$$\rho(t,x) = \frac{1}{2^d\,\pi^{\frac{d+1}{2}}\,\Gamma\left(\frac{d-1}{2}\right)} \int_0^\infty \int_0^\pi k^d\, \cos(k\,(t - x\,\cos\phi))\, \sin^{d+2}\phi\; d\phi\, dk$$
$$= \frac{1}{2^{\frac{d}{2}+1}\,\pi^{\frac{d}{2}}} \int_0^\infty k^d\, \left(k^2\,x^2\right)^{\frac{2-d}{4}}\, J_{\frac{d}{2}-1}\left(\sqrt{k^2\,x^2}\right)\, \cos(k\,t)\; dk$$
$$= \frac{1}{2^{\frac{d}{2}+1}\,\pi^{\frac{d}{2}}} \Re \int_0^\infty k^d\, \left(k^2\,x^2\right)^{\frac{2-d}{4}}\, J_{\frac{d}{2}-1}\left(\sqrt{k^2\,x^2}\right)\, e^{-\epsilon\,k + i\,k\,t}\; dk \tag{5.2}$$

A small positive imaginary part $i\epsilon$ is added to t to guarantee the k integral convergences. Continuing,

$$= \frac{1}{2\pi^{\frac{d+1}{2}}} \Re\left\{ \frac{\left(d\,(\epsilon - i\,t)^2 - x^2\right)}{(\epsilon - i\,t)^{d+3}} \left(1 + \frac{x^2}{(\epsilon - i\,t)^2}\right)^{-\frac{d+3}{2}} \Gamma\left(\frac{d+1}{2}\right) \right\}$$
$$= -\frac{1}{2\pi^{\frac{d+1}{2}}} \frac{(d\,t^2 + x^2)}{(t^2 - x^2)^{\frac{d+3}{2}}} \Gamma\left(\frac{d+1}{2}\right)\, \sin\left(\frac{d\,\pi}{2}\right) \tag{5.3}$$

The $\sin(d\pi/2)$ factor shows this result only holds for odd d. Restricting ourselves to odd d's, the final result for the point separated energy density in Minkowski space is

$$\rho(t,x) = -\frac{(-1)^{\frac{d-1}{2}}\,\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{\frac{d+1}{2}}} \frac{(d\,t^2 + x^2)}{(t^2 - x^2)^{\frac{d+3}{2}}} \tag{5.4}$$

Now we consider the correlation function (2.25). By identifying the two point function derivatives

$$G_{x_\perp x_\perp}(t,x) = -\left\langle 0 \left| \nabla_{x_\perp}^2 \left(\hat{\phi}(t_1,\mathbf{x}_1)\hat{\phi}(t_2,\mathbf{x}_2)\right)\right| 0 \right\rangle$$
$$= \frac{1}{2^d\,\pi^{\frac{d+1}{2}}\,\Gamma\left(\frac{d-1}{2}\right)} \int_0^\infty \int_{-\infty}^\infty \frac{k_\perp^d\, e^{-i\left(x\,k_x - t\,\sqrt{k_\perp^2 + k_x^2}\right)}}{\sqrt{k_\perp^2 + k_x^2}}\, dk_x\, dk_\perp \tag{5.5a}$$

$$G_{xx}(t,x) = -\left\langle 0 \left| \frac{\partial^2}{\partial x^2} \left(\hat{\phi}(t_1,\mathbf{x}_1)\hat{\phi}(t_2,\mathbf{x}_2)\right)\right| 0 \right\rangle$$

$$= \frac{1}{2^d\,\pi^{\frac{d+1}{2}}\,\Gamma\left(\frac{d-1}{2}\right)} \int_0^\infty \int_{-\infty}^\infty \frac{k_\perp^{d-2}\,k_x^2}{\sqrt{k_\perp^2+k_x^2}} e^{-i\left(x\,k_x - t\sqrt{k_\perp^2+k_x^2}\right)}\,dk_x\,dk_\perp \qquad (5.5b)$$

$$G_{tx}(t,x) = \left\langle 0 \left| \frac{\partial^2}{\partial t\,\partial x}\left(\hat\phi(t_1,\mathbf{x}_1)\hat\phi(t_2,\mathbf{x}_2)\right)\right| 0 \right\rangle$$

$$= \frac{1}{2^d\,\pi^{\frac{d+1}{2}}\,\Gamma\left(\frac{d-1}{2}\right)} \int_0^\infty \int_{-\infty}^\infty k_\perp^{d-2}\,k_x\,e^{-i\left(x\,k_x - t\sqrt{k_\perp^2+k_x^2}\right)}\,dk_x\,dk_\perp \qquad (5.5c)$$

we write the energy density correlation function as

$$\Delta\rho^2(t,x) = \frac{d\,G^2_{x_\perp x_\perp}(t,x)}{2\,(d-1)} + G_{x_\perp x_\perp}(t,x)\,G_{xx}(t,x) + G^2_{xx}(t,x) + G^2_{tx}(t,x) \qquad (5.6)$$

We proceed with the evaluation of the Green functions in a similar manner as we did for the point separated energy density above and obtain

$$G_{x_\perp x_\perp}(t,x) = \frac{(-1)^{\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2}\right)}{2\,\pi^{\frac{d+1}{2}}}\frac{(d-1)}{(t^2-x^2)^{\frac{d+1}{2}}} \qquad (5.7a)$$

$$G_{xx}(t,x) = \frac{(-1)^{\frac{d+1}{2}}\,\Gamma\left(\frac{d+1}{2}\right)}{2\,\pi^{\frac{d+1}{2}}}\frac{(t^2+d\,x^2)}{(t^2-x^2)^{\frac{d+3}{2}}} \qquad (5.7b)$$

$$G_{tx}(t,x) = \frac{(-1)^{\frac{d+1}{2}}\,\Gamma\left(\frac{d+1}{2}\right)}{2\,\pi^{\frac{d+1}{2}}}\frac{(d+1)\,t\,x}{(x^2-t^2)^{\frac{d+3}{2}}} \qquad (5.7c)$$

With these results, correlation function is

$$\Delta\rho^2(t,x) = \frac{\Gamma\left(\frac{d+1}{2}\right)^2}{\pi^{d+1}}\left(\frac{4\,t^2\,x^2 + d\,(t^2+x^2)^2}{(t^2-x^2)^{d+3}}\right) \qquad (5.8)$$

The constant χ_d is now a function of the temporal and spatial separation,

$$\chi_d(t,x) = \frac{d+1}{2}\left(\frac{4\,t^2\,x^2 + d\,(t^2+x^2)^2}{(d\;t^2+x^2)^2}\right), \qquad (5.9)$$

and we write the correlation function in terms of the square of the point separated energy density

$$\Delta\rho^2(t,x) = \chi_d(t,x)\,(\rho(t,x))^2 \qquad (5.10)$$

Our dimensionless measure is also a function of the separation:

$$\Delta(t,x) = \frac{(d+1)\,\left(d\,(t^2+x^2)^2 + 4\,t^2\,x^2\right)}{(d+1)\,\left(d\,(t^2+x^2)^2 + 4\,t^2\,x^2\right) + 2\,(d\,t^2+x^2)^2} \tag{5.11}$$

To extract physical meaning out of this for a point-wise quantum field theory, we have to work in the $(t,x) \to 0$ limit (recall $t = t_1 - t_2$, $\mathbf{x} = \mathbf{x_1} - \mathbf{x_2} = x\hat{x}$), for only then $\rho(t,x) \to \langle 0\,|\hat{\rho}|\,0\rangle$. With this in mind, we parameterize the direction dependence via $t = r\sin\theta$ and $x = r\cos\theta$ (note this is only a parameterization, the imaginary time τ and x shown below would carry physical meaning in the Euclidean sense). In the $r \to 0$ limit we have

$$\Delta(\theta) = \frac{(d+1)\,\left(1+2\,d-\cos(4\,\theta)\right)}{(d+1)\,\left(1+2\,d-\cos(4\,\theta)\right) + 4\left(\cos^2(\theta) + d\,\sin^2(\theta)\right)^2} \tag{5.12}$$

which, as expected is finite. Taking the limit along the $t-$axis ($\theta = \pi/2$), we get

$$\Delta(t, x=0) = \frac{1+d}{1+3\,d} = \Delta_{\text{Minkowski}} \tag{5.13}$$

On the other hand, taking the limit along the spatial direction:

$$\Delta(t=0, x) = \frac{d\,(d+1)}{2+d+d^2} = \Delta_{L,\text{Reg}} \tag{5.14}$$

We also approach this problem in another way. Since both the point separated energy density and the correlation function have a direction dependence, we "average" over the direction. We take the hyper-spherical averaging procedure. This involves first Wick rotating to imaginary time ($t \to i\tau$). Then we take the hyper-spherical average in the Euclidean geometry and then Wick rotate back to Minkowski space. For the energy density

$$\rho_E(\tau,x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{2\,\pi^{\frac{d+1}{2}}}\frac{(d\,\tau^2 - x^2)}{(\tau^2+x^2)^{\frac{d+3}{2}}} \tag{5.15}$$

Now expressing $\tau = r\sin\theta$ and $x = r\cos\theta$ we do the averaging

$$\begin{aligned}\rho_E(r) &= \frac{1}{2\pi}\int_0^{2\pi} \rho_E(r\sin\theta, r\cos\theta)\,d\theta \\ &= \frac{\Gamma\left(\frac{d+1}{2}\right)}{4\,\pi^{\frac{d+3}{2}}\,r^{d+1}}\int_0^{2\pi}\left(d\sin(\theta)^2 - \cos(\theta)^2\right)d\theta \\ &= \frac{(d-1)\Gamma\left(\frac{d+1}{2}\right)}{4\,\pi^{\frac{d-1}{2}}\,r^{d+1}}\end{aligned} \tag{5.16}$$

W do the same for the correlation function:

$$\begin{aligned}
\Delta\rho_E^2(r) &= \frac{(d+1)\Gamma\left(\frac{d+1}{2}\right)^2}{32\,\pi^{d+2}\,r^{2(d+1)}} \int_0^{2\pi} (d-1+(d+1)\,\cos(4\,\theta))\,d\theta \\
&= \frac{(d^2-1)\,\Gamma\left(\frac{d+1}{2}\right)^2}{32\,\pi^{d+1}\,r^{2\,(d+1)}}
\end{aligned} \tag{5.17}$$

With these results, we have

$$\chi_{d,\mathrm{Avg}} = \frac{1+d}{2\,(d-1)} \quad \text{and} \quad \Delta_{d,\mathrm{Avg}} = \frac{1+d}{3d-1} \tag{5.18}$$

independent of whether or not we Wick rotate back to Minkowski space. Also,

d	1	3	5	∞
Δ_{Avg}	1	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{1}{3}$

It is interesting to observe that the first set of results depend on the direction the two points come together, and changes if one averages over all directions. This feature of point-separation is known, but it could also reveal some properties of possible extended structure of the underlying spacetime.

VI. DISCUSSIONS

Let us ponder on the implication of these findings pertaining to a) fluctuations to mean ratio and the validity of semiclassical gravity b) the dependence of fluctuations on both the intrinsic scale (defined by smearing or point-separation) and the extrinsic scale (such as the Casimir or finite temperature periodicity) c) the treatment of divergences and meaning of regularization

A. Fluctuation to Mean ratio and Validity of SCG

If we adopt the criterion of Kuo and Ford [4] that the variance of the fluctuation relative to the mean-squared (vev taken with respect to the ordinary Minkowskian vacuum) being of the order unity be an indicator of the failure of SCG, then all spacetimes studied above would indiscriminately fall into that category, and SCG fails wholesale, regardless of the scale these physical quantities are probed. This contradicts with the common expectation that SCG is valid at scales below Planck energy. We believe the criterion for the validity or failure of a theory should depend on the range or the energy probed. Our findings here suggest that this is indeed the case: Both the mean (the vev of EMT) with respect to the Minkowski vacuum) AND the fluctuations of EMT increase as the scale deceases. As one probes into an increasingly finer scale or higher energy the expectation value of EMT grows in value and the induced metric fluctuations become important, leading to the failure of SCG. A generic scale for this to happen is the Planck length. At such energy densities and above, particle creation from the quantum field vacuum would become copious and their backreaction on the background spacetime would become important [8]. Fluctuations in the quantum field EMT entails these quantum processes. The induced metric fluctuations [10,11] renders the smooth manifold structure of spacetime inadequate, spacetime foams [20] including topological transitions [21] begin to appear and SCG no longer can provide an adequate description of these dominant processes. This picture first conjured by Wheeler is consistent with the common notion adopted in SCG, and we believe it is a valid one.

B. Dependence of fluctuations on intrinsic and extrinsic scales

In the previous section we have presented some detailed analysis on the results of our calculations for the fluctuations of the energy density for the separate cases of Minkowski and Casimir states. Let us now look at the bigger picture and see if we can capture the essence of these results with some general qualitative arguments. We want to see if there is any simple reason behind the following results we obtained:

a) $\Delta = O(1)$

b) the specific numeric value of Δ for the different cases

c) why Δ for the Minkowski space from the coincidence limit of taking a spatial point separation is identical to the Casimir case at the coincidence limit (6/7) and identical to the hot flat space result (2/5) [16] from taking the coincidence limit of a temporal point separation?

Point a) has also been shown by earlier calculations [4,5], and our understanding is that this is true only for states of quantum nature, including the vacuum and certain squeezed states, but probably not true for states of a more classical nature like the coherent state. We also emphasized that this result should not be used as a criterion for the validity of semiclassical gravity.

For point b), we can trace back the calculation of the fluctuations (second moment) of the energy momentum tensor in ratio to its mean (first moment) to the integral of the term containing the inner product of two momenta $\mathbf{k}_1 \cdot \mathbf{k}_2$ summed over all participating modes. The modes contributing to this are different for different geometries, e.g., Minkowski versus Casimir boundary –for the Einstein universe this enters as 3j symbols – and could account for the difference in the numerical values of Δ for the different cases.

For point c) the difference of results between taking the coincidence limit of a spatial versus a temporal point separation is well-known in QFTCST. The case of temporal split involves integration of three spatial dimensions while the case of spatial split involve integration of two remaining spatial and one temporal dimension, which would give different results. The calculation of fluctuations involves the second moment of the field and is in this regard similar to what enters into the calculation of moments of inertia [13] for rotating objects. We suspect that the difference between the temporal and the spatial results is similar, to the extent this analogy holds, to the difference in the moment of inertia of the same object but taken with respect to different axes of rotation.

It may appear surprising, as we felt when we first obtained these results, that in a Minkowski calculation the result of Casimir geometry or thermal field should appear, as both cases involve a scale – the former in the spatial dimension and the latter in the (imaginary) temporal dimension. But if we note that the results for Casimir geometry or thermal field are obtained at the coincidence (ultraviolet) limit, when the scale (infrared) of the problem does not intercede in any major way, then the main components of the calculations for these two cases would be similar to the two cases of taking coincident limit in the spatial and temporal directions in Minkowski space. All of these cases are effectively devoid of scale as far as the pointwise field theory is concerned. As soon as we depart from this limit the effect of the presence of a scale shows up. The point-separated or field-smeared results for the Casimir calculation in Sec. 4 shows clearly that the boundary scale enters in a major way and the result for the fluctuations and the ratio are different from those obtained at the coincident limit. For other cases where a scale enters intrinsically in the problem such as that of a massive or non-conformally coupled field it would show a similar effect in these regards as the present cases (of Casimir and thermal field) where a periodicity condition exists (in the spatial and temporal directions respectively). We expect a similar strong disparity between point-coincident and point-separated cases. The field theory changes its nature in a fundamental and physical way when this limit is taken. This brings us to the second major issue brought out in this investigation, i.e., the appearance of divergences and the meaning of regularization in the light of a point-separated versus a point-defined quantum field theory.

C. Regularization in the Fluctuations of EMT

From our calculations, the smeared energy density fluctuations for the Casimir topology has the form

$$\Delta\rho_L^2(\sigma) = \Delta\rho_L^{\rm div} + \Delta\rho_L^{\rm cross} + \Delta\rho_L^{\rm fin} \tag{6.1}$$

with

$$\Delta\rho_L^{\rm div} = \chi_d \left(\rho_L^{\rm div}\right)^2 = \chi_d(\rho(\sigma))^2 \tag{6.2a}$$

$$\Delta\rho_L^{\rm cross} = 2\chi_d \rho_L^{\rm div}\rho_L^{\rm fin} \tag{6.2b}$$

$$\Delta\rho_L^{\rm fin} = 2\chi_{d,L}\left(\rho_L^{\rm fin}\right)^2 + \text{terms that vanish as } \sigma \to 0 \tag{6.2c}$$

where χ_d is the ratio between the fluctuations for Minkowski space and the square of the corresponding energy density: $\Delta\rho^2 = \chi_d(\rho(\sigma))^2$. Our results show that $\Delta\rho_L^2(\sigma)$ diverges as the width σ of the smearing function shrinks to zero with contributions from the truly divergent and the cross terms. We also note that the divergent term $\Delta\rho^{\rm div}$ is state independent, in the sense that it is independent of L, while the cross term $\Delta\rho^{\rm cross}$ is state dependent, as is the finite term $\Delta\rho^{\rm fin}$.

If we want to ask about the strength of fluctuations of the energy density, the relevant quantity to study is the energy density correlation function $H(x,y) = \langle\hat{\rho}(x)\hat{\rho}(y)\rangle - \langle\hat{\rho}(x)\rangle\langle\hat{\rho}(y)\rangle$. It is finite at $x \neq y$ for a linear quantum theory (this happens since the divergences for $\langle\hat{\rho}(x)\hat{\rho}(y)\rangle$ are exactly the same as the product $\langle\hat{\rho}(x)\rangle\langle\hat{\rho}(y)\rangle$), but diverges as $y \to x$, corresponding to the coincident or unsmeared limit $\sigma \to 0$.

To define a procedure for rendering our expression for $\Delta\rho_L^2(\sigma)$ finite, one can see that there exists choices – which means ambiguities in the regularization scheme. Three possibilities present themselves: The first is to just drop the state independent $\Delta\rho^{\rm div}$. This is easily seen to fail since we are left with the divergences from the cross term. The second is to neglect all terms that diverge as $\sigma \to 0$. This is too rash a move since $\Delta\rho^{\rm cross}$ has, along with its divergent parts, ones that are finite in the $\sigma \to 0$ limit. This comes about since it is of the form $\rho_L^{\rm div}\rho_L^{\rm fin}$ and the negative powers of σ present in $\rho_L^{\rm div}$ will cancel out against the positive powers in $\rho_L^{\rm fin}$. Besides, they yield results in disagreement with earlier results using well-tested methods such as normal ordering in flat space [4] and zeta-function regularization in curved space [5].

The third choice is the one we have used in this paper. For the energy density, we can think of regularization as computing the contribution "above and beyond" the Minkowski vacuum contribution. Same for regularizing the fluctuations. So we need to first determine for Minkowski space vacuum how the fluctuations of the energy density are related to the vacuum energy density $\Delta\rho^2 = \chi\ (\rho)^2$. This we obtained for finite smearing. For Casimir topology the sum of the divergent and cross terms take the form

$$\Delta\rho_L^{\rm div} + \Delta\rho_L^{\rm cross} = \chi\left\{\left(\rho_L^{\rm div}\right)^2 + 2\rho_L^{\rm div}\rho_L^{\rm fin}\right\} = \chi\left\{(\rho_L)^2 - \left(\rho_L^{\rm fin}\right)^2\right\} \tag{6.3}$$

where χ is the ratio derived for Minkowski vacuum. We take this to represent the (state dependent) vacuum contribution. What we find interesting is that to regularize the smeared energy density fluctuations, a state dependent subtraction must be used. With this, just the $\sigma \to 0$ limit of the finite part $\Delta\rho_L^{\text{fin}}$ is identified as the regularized fluctuations $\Delta\rho^2_{L,\text{Reg}}$. The ratio χ_L thus obtained gives exactly the same result as derived by Kou and Ford for $d = 3$ via normal ordering and by ourselves for arbitrary d via the ζ-function.

That this procedure is the one to follow can be seen by considering the problem from the point separation method. For this method, the energy density expectation value is defined as the $x' \to x$ limit of

$$\rho(x, x') = \mathcal{D}_{x,x'} G(x, x') \tag{6.4}$$

for the suitable Green function $G(x, x')$ and $\mathcal{D}_{x,x'}$ is a second order differential operator. (For the more general stress tensor, details are reviewed in [16].) In the limit $x' \to x$, $G(x, x')$ is divergent. The Green function is regularized by subtracting from it a Hadamard form $G^L(x, x')$: $G_{\text{Reg}}(x, x') = G(x, x') - G^L(x, x')$ [17]. With this, the regularized energy density can be obtained

$$\rho_{\text{Reg}}(x) = \lim_{x' \to x} \left(\mathcal{D}_{x,x'} G_{\text{Reg}}(x, x')\right) \tag{6.5}$$

Or, re-arranging terms, we can define the divergent and finite pieces as

$$G^{\text{div}}(x, x') = G^L(x, x'), \quad G^{\text{fin}}(x, x') = G_{\text{Reg}}(x, x') = G(x, x') - G^L(x, x') \tag{6.6}$$

and

$$\rho(x, x') = \rho^{\text{div}}(x, x') + \rho^{\text{fin}}(x, x') \tag{6.7}$$

$$\rho^{\text{div}}(x, x') = \mathcal{D}_{x,x'} G^{\text{div}}(x, x') \quad \text{and} \quad \rho^{\text{fin}}(x, x') = \mathcal{D}_{x,x'} G^{\text{fin}}(x, x')$$

so that $\rho_{\text{Reg}}(x) = \lim_{x' \to x} \rho^{\text{fin}}(x, x')$, which corresponds to the $\sigma \to 0$ limit in our computation of the Casimir energy density.

Now turning to the fluctuations, we have the point separated expression for the correlation function

$$H(x,y) = \lim_{x'\to x} \lim_{y'\to y} \mathcal{D}_{x,x'}\mathcal{D}_{y,y'} G(x,x',y,y') \tag{6.8}$$

where $G(x,x',y,y')$ is the suitable four point function. For linear theories we use Wick's Theorem to express this in terms of products of Green functions $G(x,x',y,y') = G(x,y)G(x',y')\ +\ \text{permutations of}(x,x',y,y')$. Excluded from the permutations is $G(x,x')G(y,y')$. (Details are in [16], which includes correct identifications of needed permutations and Green functions.) The general form is

$$H(x,y) = \lim_{x'\to x} \lim_{y'\to y} \mathcal{D}_{x,x'}\mathcal{D}_{y,y'} G(x,y)G(x',y')\ +\ \text{permutations} \tag{6.9}$$

The $(x',y') \to (x,y)$ limits are only retained to keep track of which derivatives act on which Green functions, but we can see there are no divergences for $y \neq x$. However, to get the point-wise fluctuations of the energy density, the divergences from $\lim_{y\to x} G(x,y)$ will present a problem. Splitting the Green function into its finite and divergent pieces, we can recognize terms leading to those we found for $\Delta\rho_L^2(\sigma)$:

$$H(x,y) = H^{\text{div}}(x,y) + H^{\text{cross}}(x,y) + H^{\text{fin}}(x,y) \tag{6.10}$$

where

$$H^{\text{div}}(x,y) = \lim_{x'\to x} \lim_{y'\to y} \mathcal{D}_{x,x'}\mathcal{D}_{y,y'} G^{\text{div}}(x,y)G^{\text{div}}(x',y') \tag{6.11a}$$

$$H^{\text{cross}}(x,y) = 2\lim_{x'\to x} \lim_{y'\to y} \mathcal{D}_{x,x'}\mathcal{D}_{y,y'} G^{\text{div}}(x,y)G^{\text{fin}}(x',y') \tag{6.11b}$$

$$H^{\text{fin}}(x,y) = \lim_{x'\to x} \lim_{y'\to y} \mathcal{D}_{x,x'}\mathcal{D}_{y,y'} G^{\text{fin}}(x,y)G^{\text{fin}}(x',y') \tag{6.11c}$$

plus permutations. Thus we see the origin of both the divergent and cross terms. When the un-regularized Green function is used, we must get a cross term, along with the expected divergent term. If the fluctuations of the energy density is regularized via point separation, i.e. $G(x,x')$ is replaced by $G_{\text{Reg}}(x,x') = G^{\text{fin}}(x,y)$, then we should do the same replacement for the fluctuations. When this is done, it is only the finite part above that will be left and we can define the point-wise fluctuations as

$$\Delta\rho^2_{\text{Reg}} = \lim_{y\to x} H^{\text{fin}}(x,y) \tag{6.12}$$

The parallel with the smeared-field derivation presented here can be seen when the analysis of $G_L(\sigma)_{,zz}$ and $G_L(\sigma)_{,x_\perp x_\perp}$ in the Appendix is considered. There it is shown they are derivatives of Green functions and can be separated into state-independent divergent part and state-dependent finite contribution: $G_L(\sigma)_{,i} = G^{\rm div}_{L,i} + G^{\rm fin}_{L,i}$, same as the split hereby shown for the Green function.

When analyzing the energy density fluctuations, discarding the divergent piece is the same as subtracting from the Green function its divergent part. If this is done, we also no longer have the cross term, just as viewing the problem from the point separation method outlined above. We feel this makes it problematic to analyze the cross term without also including the divergent term. At the same time, regularization of the fluctuations involving the subtraction of state dependent terms as realized in this calculation raises new issues on regularization which merits further investigations.

To end this discussion, we venture one philosophical point we find resounding throughout all the cases studied here. It has to do with the meaning of a point-defined versus a point-separated field theory, the former we take as an effective theory coarse-grained from the latter, the point-separated theory reflecting a finer level of spacetime structure. It bears on the meaning of regularization, not just at the level of technical procedures, but related to finding an effective description and matching with physics observed at a coarser scale or lower energy.

In particular, we feel that finding a finite energy momentum tensor (and its fluctuations as we do here) which occupied the center of attention in the research of quantum field theory in curved spacetime in the 70's is only a small part of a much larger and richer structure of theories of fields and spacetimes . We come to understand that whatever regularization method one uses to get these finite parts in a point-wise field theory should not be viewed as universally imparting meaning beyond its specified function, i.e., to identify the divergent pieces and provide a prescription for their removal. We believe the extended structure of spacetime (e.g., via point-separation or smearing) and the field theory defined therein has its own much fuller meaning beyond just reproducing the well-recognized result in ordinary quantum field theory as we take the point-wise or coincident limit. In this way of thinking, the divergence- causing terms are only 'bad' when they are forced to a point-wise limit, because of our present inability to observe or resolve otherwise . If we accord them with the full right of existence beyond this limit, and acknowledge that their misbehavior is really due to our own inability to cope, we will be rewarded with the discovery of new physical phenomena and ideas of a more intricate world. (Maybe this is just another way to appreciate the already well-heeded paths of string theory.)

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APPENDIX A: EVALUATION OF TWO POINT FUNCTIONS FOR CASIMIR TOPOLOGY

We want to compute the smeared derivatives of the field operator two-point functions for the Casimir geometry:

$$G_L(\sigma)_{,x_\perp x_\perp} = \left\langle 0_L \left| \left((\nabla_\perp \phi)\,(f_{\mathbf{x}}) \right)^2 \right| 0_L \right\rangle \quad \text{and} \quad G_L(\sigma)_{,zz} = \left\langle 0_L \left| \left((\partial_z \phi)\,(f_{\mathbf{x}}) \right)^2 \right| 0_L \right\rangle , \tag{A1}$$

where $|\,0_L\rangle$ is the Casimir vacuum. Performing the differentiation and taking the vacuum expectation values, we need the integrals and sums

$$G_L(\sigma)_{,x_\perp x_\perp} = \frac{l}{2^d\,\pi^{\frac{d+1}{2}}\,\Gamma\!\left(\frac{d-1}{2}\right)} \sum_{n=-\infty}^{\infty} \int_0^\infty \frac{k^d}{\sqrt{k^2 + l^2\,n^2}} e^{-2\left(k^2 + l^2\,n^2\right)\sigma^2}\,dk \tag{A2a}$$

$$G_L(\sigma)_{,zz} = \frac{l}{2^d\,\pi^{\frac{d+1}{2}}\,\Gamma\!\left(\frac{d-1}{2}\right)} \sum_{n=-\infty}^{\infty} \int_0^\infty \frac{k^{d+2}\,l^2\,n^2}{\sqrt{k^2+l^2\,n^2}} e^{-2\left(k^2+l^2\,n^2\right)\sigma^2}\,dk \tag{A2b}$$

With the definitions of the functions

$$F_{x_\perp x_\perp}(n) = \int_0^\infty \frac{2\,k^d}{\sqrt{k^2+l^2\,n^2}} e^{-2\left(k^2+l^2\,n^2\right)\sigma^2}\,dk \tag{A3a}$$

$$F_{zz}(n) = \int_0^\infty \frac{2\,k^{d-2}\,l^2\,n^2}{\sqrt{k^2+l^2\,n^2}} e^{-2\left(k^2+l^2\,n^2\right)\sigma^2}\,dk \tag{A3b}$$

we use the Euler-Maclauren sum formula to re-arrange the terms to the more useful form ($i = x_\perp x_\perp$ or zz):

$$\begin{aligned} G_L(\sigma)_{,i} &= \frac{l}{2^d\,\pi^{\frac{d+1}{2}}\,\Gamma\!\left(\frac{d-1}{2}\right)} \lim_{N\to\infty} \left(\frac{1}{2}F_i(0) + \sum_{n=1}^{N} F_i(n)\right) \\ &= \frac{l}{2^d\,\pi^{\frac{d+1}{2}}\,\Gamma\!\left(\frac{d-1}{2}\right)} \lim_{N\to\infty} \left(\int_0^N F_i(n)\,dn + \frac{1}{2}F_i(N) + \sum_{p=1}^{q} \frac{B_{2p}}{(2p)!}\left(F_i^{(2p-1)}(N) - F_i^{(2p-1)}(0)\right)\right) \end{aligned} \tag{A4}$$

As we will show, $F_i(N)$ vanishes exponentially with N so that $F_i(N)$ and $F_i^{(2p-1)}(N)$ give no contributions to the final result and we are left with

$$G_L(\sigma)_{,i} = \frac{l}{2^d\,\pi^{\frac{d+1}{2}}\,\Gamma\!\left(\frac{d-1}{2}\right)} \left(\int_0^\infty F_i(n)\,dn - \sum_{p=1}^{q} \frac{B_{2p}}{(2p)!} F_i^{(2p-1)}(0)\right) \tag{A5}$$

This re-arrangement of the terms has allowed us to separate the expectation values into terms that diverge as $\sigma \to 0$ and those that are finite in this limit: $G_L(\sigma)_{,i} = G_{L,i}^{\text{div}} + G_{L,i}^{\text{fin}}$ with

$$G_{L,i}^{\text{div}} = \frac{l}{2^d\,\pi^{\frac{d+1}{2}}\,\Gamma\!\left(\frac{d-1}{2}\right)} \int_0^\infty F_i(n)\,dn \tag{A6a}$$

$$G_{L,i}^{\text{fin}} = -\frac{l}{2^d\,\pi^{\frac{d+1}{2}}\,\Gamma\!\left(\frac{d-1}{2}\right)} \sum_{p=1}^{q} \frac{B_{2p}}{(2p)!} F_i^{(2p-1)}(0) \tag{A6b}$$

The k integrations give the explicit form of the functions $F_i(n)$:

$$\begin{aligned} F_{x_\perp x_\perp}(n) = {} & \frac{l^d\,n^d\,\Gamma\!\left(-\frac{d}{2}\right)\,\Gamma\!\left(\frac{d+1}{2}\right)}{\sqrt{\pi}}\, {}_1F_1\left(\frac{1}{2}; 1+\frac{d}{2}; -2\,l^2\,n^2\,\sigma^2\right) \\ & + \frac{\Gamma\!\left(\frac{d}{2}\right)}{2^{\frac{d}{2}}\,\sigma^d}\, {}_1F_1\left(\frac{1-d}{2}; 1-\frac{d}{2}; -2\,l^2\,n^2\,\sigma^2\right) \end{aligned} \tag{A7a}$$

and

$$F_{zz}(n) = \frac{l^d\, n^d\, \Gamma\left(1-\frac{d}{2}\right)\, \Gamma\left(\frac{d-1}{2}\right)}{\sqrt{\pi}}\, {}_1F_1\left(\frac{1}{2};\frac{d}{2};-2\,l^2\,n^2\,\sigma^2\right) + \frac{l^2\,n^2}{2^{\frac{d}{2}-1}\,\sigma^{d-2}}\Gamma\left(\frac{d}{2}-1\right)\, {}_1F_1\left(\frac{3}{2}-\frac{d}{2};2-\frac{d}{2};-2\,l^2\,n^2\,\sigma^2\right) \tag{A7b}$$

with ${}_1F_1\,(a;b;z)$ the Kummer confluent hypergeometric function.

We carry out the n integrations and obtain the divergent parts of the expectation values

$$G^{\rm div}_{L,x_\perp x_\perp} = \frac{(d-1)\,\Gamma\left(\frac{d+1}{2}\right)}{2^{\frac{3\,(d+1)}{2}}\,d\,\pi^{\frac{d}{2}}\,\Gamma\left(\frac{d}{2}\right)\,\sigma^{d+1}} \tag{A8a}$$

$$G^{\rm div}_{L,zz} = \frac{\Gamma\left(\frac{d+1}{2}\right)}{2^{\frac{3\,(d+1)}{2}}\,d\,\pi^{\frac{d}{2}}\,\Gamma\left(\frac{d}{2}\right)\,\sigma^{d+1}} \tag{A8b}$$

Turning to the finite contribution, we need the general form of

$$H = \left.\frac{d^{2p-1}}{dn^{2p-1}}\left(A\,n^{\beta}{}_1F_1\left(\alpha;\gamma;-2\,l^2\,n^2\,\sigma^2\right)\right)\right|_{n=0} \tag{A9}$$

To this end, we make use of the relation

$$\left.\frac{d^p g\left(an^2\right)}{dn^p}\right|_{n=0} = \begin{cases} (p-1)!!\,(2a)^{\frac{p}{2}}\,g^{\left(\frac{p}{2}\right)}(0); & p \text{ even} \\ 0; & p \text{ odd} \end{cases} \tag{A10}$$

along with

$$\begin{aligned} \frac{d\,{}_1F_1\left(\alpha,\gamma;z\right)}{dz} &= \frac{\alpha}{\gamma}{}_1F_1\left(\alpha+1,\gamma+1;z\right) \\ \Rightarrow \frac{d^p\,{}_1F_1\left(\alpha,\gamma;z\right)}{dz^p} &= \frac{\Gamma(\alpha+p)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma+p)}{}_1F_1\left(\alpha+p,\gamma+p;z\right) \end{aligned} \tag{A11}$$

For γ not a negative integer, ${}_1F_1\,(\alpha,\gamma;0) = 1$. These results lead to

$$\begin{aligned} H &= \frac{A\,(2p-1)!}{\Gamma(2p-\beta)}\left.\left(\frac{d^{2p-1-\beta}{}_1F_1\left(\alpha;\gamma;-2\,l^2\,n^2\,\sigma^2\right)}{dn^{2p-1-\beta}}\right)\right|_{n=0} \\ &= \begin{cases} \frac{A\,(2p-1)!}{\Gamma(2p-\beta)}2^{2p-\beta-1}\left(-\left(l^2\,\sigma^2\right)\right)^{\frac{2p-\beta-1}{2}}\,(2p-\beta-2)!!\left.\left(\frac{d^{\frac{2p-\beta-1}{2}}{}_1F_1(\alpha;\gamma;z)}{dz^{\frac{2p-\beta-1}{2}}}\right)\right|_{z=0}; & \beta \quad \text{odd} \\ 0; & \beta \quad \text{even} \end{cases} \end{aligned} \tag{A12}$$

Staying with β odd, the final result is

$$\frac{A\ (-1+2\,p)!}{\Gamma(-\beta+2\,p)} 2^{2p-\beta-1} \left(-\left(l^2\,\sigma^2\right)\right)^{\frac{2p-\beta-1}{2}}\ (2p-\beta-2)!! \frac{\Gamma(\gamma)\ \Gamma\left(-\frac{1}{2}+\alpha-\frac{\beta}{2}+p\right)}{\Gamma(\alpha)\ \Gamma\left(-\frac{1}{2}-\frac{\beta}{2}+\gamma+p\right)} \tag{A13}$$

To determine the finite contributions to the smeared Green function derivatives, we use

		A	β	α	γ
$F_{x_\perp x_\perp}$	term 1	$l^d\,\Gamma\left(-\frac{d}{2}\right)\ \Gamma\left(\frac{d+1}{2}\right)/\sqrt{\pi}$	d	$\frac{1}{2}$	$1+\frac{d}{2}$
$F_{x_\perp x_\perp}$	term 2	$2^{-\frac{d}{2}}\,\sigma^{-d}\Gamma\left(\frac{d}{2}\right)$	0	$\frac{1-d}{2}$	$1-\frac{d}{2}$
F_{zz}	term 1	$l^d\,\Gamma\left(1-\frac{d}{2}\right)\ \Gamma\left(\frac{d-1}{2}\right)/\sqrt{\pi}$	d	$\frac{1}{2}$	$\frac{d}{2}$
F_{zz}	term 2	$2^{1-\frac{d}{2}}\,l^2\,\sigma^{2-d}\,\Gamma\left(-1+\frac{d}{2}\right)$	2	$\frac{3}{2}-\frac{d}{2}$	$2-\frac{d}{2}$

For d odd, only the first terms of $F_{x_\perp x_\perp}$ and F_{zz} contribute. For d even, the situation involves more analysis, since for the second terms, even though β is even, γ is a negative integer. This implies ${}_1F_1\left(\alpha,\gamma,-2l^2n^2\sigma^2\right)$ divergent structure needs to be considered as well. For d odd,

$$G^{\rm fin}_{L,x_\perp x_\perp} = -\frac{l^{d-1}\,d\ (d-1)\ \Gamma\left(-\frac{d}{2}\right)\ \Gamma\left(\frac{d}{2}\right)}{2^{d+3}\,\pi^{\frac{d+3}{2}}} \sum_{p=1} \left(-4l^2\right)^p \sigma^{2\,(p-1)} p\ (2p-1)\ \frac{B_{2p+d-1}\ (2p-3)!!\,\Gamma\left(p-\frac{1}{2}\right)}{(2p+d-1)\ (2p)!\,\Gamma\left(p+\frac{d}{2}\right)} \tag{A14a}$$

and

$$G^{\rm fin}_{L,zz} = \frac{l^{d-1}\,d\,\Gamma\left(-\frac{d}{2}\right)\ \Gamma\left(\frac{d}{2}\right)}{2^{d+3}\,\pi^{\frac{d+3}{2}}} \sum_{p=1}^{\infty} \left(-4l^2\right)^p \sigma^{2\,(p-1)} p\ (2p-1)\ (2p+d-2)\ \frac{B_{2p+d-1}\ (2p-3)!!\,\Gamma\left(p-\frac{1}{2}\right)}{(2p+d-1)\ (2\,p)!\,\Gamma\left(p+\frac{d}{2}\right)} \tag{A14b}$$

APPENDIX B: PLOTTING SMEARED CASIMIR RESULTS

In this appendix we outline how Δ_L and $\Delta_{L,\rm Reg}$ are manipulated so they can be plotted as a function of σ, the smearing width. We know $G_L(\sigma)_{,i} = G^{\rm div}_{L,i} + G^{\rm fin}_{L,i}$ and $\rho = \rho^{\rm div} + \rho^{\rm fin}$, where

$$\rho^{\rm div} = \frac{1}{32\,\pi^2\,\sigma^4}, \quad G^{\rm div}_{L,x_\perp x_\perp} = \frac{1}{48\,\pi^2\,\sigma^4} = \frac{2}{3}\rho^{\rm div} \quad \text{and} \quad G^{\rm div}_{L,zz} = \frac{1}{96\,\pi^2\,\sigma^4} = \frac{1}{3}\rho^{\rm div} \tag{B1}$$

We write $G_L(\sigma)_{,i} = F_i(0) + 2\sum_{n=1}^{\infty} F_i(n)$ and $\rho_L(\sigma) = F_\rho(0) + 2\sum_{n=1}^{\infty} F_\rho(n)$ with

$$\begin{aligned}
F_1 &= \frac{n}{8\, e^{8\,n^2\,\pi^2\,\sigma^2}\,\sigma^2} + \frac{\mathrm{Erfc}\left(2\,\sqrt{2}\,n\,\pi\,\sigma\right)}{32\,\sqrt{2\,\pi}\,\sigma^3} - \frac{n^2\,\pi^{\frac{3}{2}}\,\mathrm{Erfc}\left(2\,\sqrt{2}\,n\,\pi\,\sigma\right)}{2\,\sqrt{2}\,\sigma} \\
F_2 &= \frac{n^2\,\pi^{\frac{3}{2}}\,\mathrm{Erfc}\left(2\,\sqrt{2}\,n\,\pi\,\sigma\right)}{2\,\sqrt{2}\,\sigma} \\
F_\rho &= \frac{n}{8\, e^{8\,n^2\,\pi^2\,\sigma^2}\,\sigma^2} + \frac{\mathrm{Erfc}\left(2\,\sqrt{2}\,n\,\pi\,\sigma\right)}{32\,\sqrt{2\,\pi}\,\sigma^3}
\end{aligned} \tag{B2}$$

The problem here is that *each* of the terms in the sum over n diverge as $\sigma \to 0$. By defining $\tilde{F}_i(n) = F_i(n)/X_i^{\mathrm{div}}$ ($i = x_\perp x_\perp, zz, \rho$ and $X_i = G^{\mathrm{div}}_{L,x_\perp x_\perp}, G^{\mathrm{div}}_{L,zz}, \rho_L^{\mathrm{div}}$) and $\tilde{X}_i = 2{\sum}'^{\infty}_{n=0}\tilde{F}_i(n) - 1$, where $\sum'$ has a factor of $\frac{1}{2}$ for $n = 0$, then

$$X_i = X_i^{\mathrm{div}}\,\tilde{X}_i \tag{B3}$$

The defined functions are

$$\begin{aligned}
\tilde{F}_{x_\perp x_\perp}(\sigma, n) &= \frac{6\,n\,\pi^2\,\sigma^2}{e^{8\,n^2\,\pi^2\,\sigma^2}} + \frac{3\,\pi^{\frac{3}{2}}\,\sigma\,\mathrm{Erfc}\left(2\,\sqrt{2}\,n\,\pi\,\sigma\right)}{2\,\sqrt{2}} - 12\,\sqrt{2}\,n^2\,\pi^{\frac{7}{2}}\,\sigma^3\,\mathrm{Erfc}\left(2\,\sqrt{2}\,n\,\pi\,\sigma\right) \\
\tilde{F}_{zz}(\sigma, n) &= 24\,\sqrt{2}\,n^2\,\pi^{\frac{7}{2}}\,\sigma^3\,\mathrm{Erfc}\left(2\,\sqrt{2}\,n\,\pi\,\sigma\right) \\
\tilde{F}_\rho(\sigma, n) &= \frac{4\,n\,\pi^2\,\sigma^2}{e^{8\,n^2\,\pi^2\,\sigma^2}} + \frac{\pi^{\frac{3}{2}}\,\sigma\,\mathrm{Erfc}\left(2\,\sqrt{2}\,n\,\pi\,\sigma\right)}{\sqrt{2}}
\end{aligned} \tag{B4}$$

Each of these new functions to be summed over are now finite as $\sigma \to 0$. We have divided out the known divergent part.

Now we can turn to the smeared fluctuations of the energy density. First, for the sum of the divergent and cross terms we have

$$\Delta\rho_L^{\mathrm{div}} + \Delta\rho_L^{\mathrm{cross}} = \frac{2}{3}\left((\rho_L^{\mathrm{div}})^2 + 2\,\rho_L^{\mathrm{div}}\,\rho_L^{\mathrm{fin}}\right) = \frac{2}{3}\,(1 + 2\tilde{\rho}_L)\,\left(\rho_L^{\mathrm{div}}\right)^2 \tag{B5}$$

We can clearly see how this has factored out the divergent coefficient. For the finite term:

$$\Delta\rho_L^{\mathrm{fin}} = \frac{(\rho_L^{\mathrm{div}})^2}{9}\left(3\,\tilde{G}^2_{L,x_\perp x_\perp} + 2\,\tilde{G}_{L,x_\perp x_\perp}\,\tilde{G}_{L,zz} + \tilde{G}^2_{L,zz}\right) \tag{B6}$$

Taking together, the fluctuations can be written as

$$\Delta\rho_L^2 = \frac{(\rho_L^{\mathrm{div}})^2}{9}\left(6 + 3\,\tilde{G}^2_{L,x_\perp x_\perp} + 2\,\tilde{G}_{L,x_\perp x_\perp}\,\tilde{G}_{L,zz} + \tilde{G}^2_{L,zz} + 12\,\tilde{\rho}_L\right) \tag{B7}$$

From this, we get the dimensionless measure

$$\Delta(\sigma, L) = \frac{6 + 3\,\tilde{G}^2_{L,x_\perp x_\perp} + 2\,\tilde{G}_{L,x_\perp x_\perp}\,\tilde{G}_{L,zz} + \tilde{G}^2_{L,zz} + 12\,\tilde{\rho}_L}{15 + 3\,\tilde{G}^2_{L,x_\perp x_\perp} + 2\,\tilde{G}_{L,x_\perp x_\perp}\,\tilde{G}_{L,zz} + \tilde{G}^2_{L,zz} + 30\,\tilde{\rho}_L + 9\,\tilde{\rho}^2_L} \tag{B8}$$

Also, considering just the finite terms

$$\Delta_{L,\mathrm{Reg}}(\sigma, L) = \frac{\Delta\rho_{L,\mathrm{Reg}}}{\Delta\rho_{L,\mathrm{Reg}} + (\rho_{L,\mathrm{Reg}})^2} = \frac{3\,\tilde{G}^2_{L,x_\perp x_\perp} + 2\,\tilde{G}_{L,x_\perp x_\perp}\,\tilde{G}_{L,zz} + \tilde{G}^2_{L,zz}}{3\,\tilde{G}^2_{L,x_\perp x_\perp} + 2\,\tilde{G}_{L,x_\perp x_\perp}\,\tilde{G}_{L,zz} + \tilde{G}^2_{L,zz} + 9\,\tilde{\rho}^2_L} \tag{B9}$$

We can now numerically evaluate the above ratios. Plots of $\Delta(\sigma, L)$ and $\Delta_{L,\mathrm{Reg}}(\sigma, L)$ as a function of σ/L are presented in Figure 1. Figure 2 presents plots of $\rho_L^{\mathrm{fin}}(\sigma, L)$ and $\sqrt{\Delta\rho_L^{\mathrm{fin}}(\sigma, L)}$.

We need to worry about the error for $\sigma \sim 1(= L)$. Considering only the periodic z direction, this was smeared with the function

$$f(z, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{z^2}{2\sigma^2}\right) \tag{B10}$$

For an error estimate, we use $f(1, \sigma)$, for as long as this is small, then the Gaussian smearing function does not detect the periodicity. At $\sigma = 0.4$, this error is only 4%.

REFERENCES

[1] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-time* (Cambridge University Press, Cambridge, England, 1989).

R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (The University of Chicago Press, Chicago, 1994).

A. A. Grib, S. G. Mamayev and V. M. Mostepanenko, *Vacuum Quantum Effects in Strong Fields* (Friedmann Laboratory Publishing, St. Petersburg, 1994).

[2] H. B. Casimir, Proc. Kon. Ned. Akad. Wet. **51**, 793 (1948).

[3] G. Barton, J. Phys. A. **24**, 991 (1991); **24** 5533 (1991).

[4] C.-I. Kou and L. Ford, Phys. Rev. D **47**, 4510 (1993). L. H. Ford, Ann. Phys. **144**, 238 (1982).

[5] N. G. Phillips and B. L. Hu, Phys. Rev. D. **55**, 6123 (1997).

[6] K. S. Thorne, *Black Holes and Time Warps* (Norton Books, 1994)

[7] B. L. Hu and N. G. Phillips, Int. J. Theor. Phys. **39** (2000) gr-qc/0004006.

[8] Ya. Zel'dovich and A. Starobinsky, Zh. Eksp. Teor. Fiz **61**, 2161 (1971) [Sov. Phys.-JETP **34**, 1159 (1971)] B. L. Hu and L. Parker, Phys. Rev. **D17**, 933 (1978). F. V. Fischetti, J. B. Hartle and B. L. Hu, Phys. Rev. **D20**, 1757 (1979). J. B. Hartle and B. L. Hu, Phys. Rev. **D20**, 1772 (1979). **21**, 2756 (1980). J. B. Hartle, Phys. Rev. D23, 2121 (1981). P. A. Anderson, Phys. Rev. D28, 271 (1983); D29, 615 (1984). E. Calzetta and B. L. Hu, Phys. Rev. **D35**, 495 (1987) A. Campos and E. Verdaguer, Phys. Rev. **D49**, 1861 (1994).

[9] J. B. Hartle and G. Horowitz, Phys. Rev. D 24, 257 (1981).

R. D. Jordan, Phys. Rev. **D36**, 3593 (1987).

C.-I. Kou and L. Ford, Phys. Rev. D **47**, 4510 (1993).

N. G. Phillips and B. L. Hu, Phys. Rev. D. **55**, 6123 (1997).

E. Flanagan and R. M. Wald, Phys. Rev. D **54**, 6233 (1996).

W. Tichy and É. É. Flanagan, Phys. Rev. **D58**, 124007 (1998).

[10] B. L. Hu, Physica **A**158, 399 (1989). B. L. Hu, Int. J. Theor. Phys. 38, 2987 (1999) gr-qc/9902064

R. Martín and E. Verdaguer, Int. J. Theor. Phys. 38, 3049 (1999) gr-qc/9812063. Phys. Lett. **B465**, 113 (1999) gr-qc/9811070 Phys. Rev. **D60**, 084008 (1999). gr-qc/9904021. gr-qc/0001098.

A. Roura and E. Verdaguer, Int. J. Theor. Phys. **38**, 3123 (1999) gr-qc/9812063

[11] E. Calzetta and B.-L. Hu, Phys. Rev. D **49**, 6636 (1994).

B. L. Hu and S. Sinha, Phys. Rev. **D 51**, 1587 (1995).

B. L. Hu and A. Matacz, Phys. Rev. **D51**, 1577 (1995).

A. Campos and E. Verdaguer, Phys. Rev. **D53**, 1927 (1996).

E. Calzetta, A. Campos and E. Verdaguer, Phys. Rev. D **56**, 2163 (1997).

F. C. Lombardo and F. D. Mazzitelli, Phys. Rev. **D55**, 3889 (1997).

A. Campos and B.-L. Hu, Phys. Rev. **D58**, 125021 (1998).

A. Campos and B.-L. Hu, Int. J. Theor. Phys. **38**, 1253 (1999).

E. Calzetta and E. Verdaguer, Phys. Rev. **D59**, 083513 (1999).

[12] See any textbook in quantum optics, e.g., D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 1994). M. O. Scully and S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge 1997). P. Meystre and M. Sargent III, *Elements of Quantum Optics* (Springer-Verlag, Berlin, 1990)

[13] We thank A. Raval for this comment.

[14] B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, 1965); Phys. Rep. **C19**, 295 (1975). S. Christensen, Phys. Rev. D14, 2490 (1976); D17, 946 (1978).

S. L. Adler, J. Lieberman and Y. J. Ng, Ann. Phys. (N.Y.) **106**, 279 (1977). R. M. Wald, Phys. Rev. **D17**, 1477 (1977).

[15] See, e.g., Zheng Yin, Phys. Lett. B466 (1999) 234. hep-th/9908152 D. Bigatti and L. Susskind, hep-th/9908056. N. Seiberg and E. Witten, hep-th/9908142

[16] N. G. Phillips, Ph. D. Thesis, University of Maryland (1999)

[17] R. M. Wald, Commun. Math. Phys. **45**, 9 (1975).

[18] C. H. Wu and L. H. Ford, Phys. Rev. D60 (1999) 104013.

[19] B. L. Hu and D. J. O'Connor, Phys. Rev. **D36**, 1701 (1987).

[20] J. A. Wheeler, Ann. Phys. (N. Y.) 2, 604 (1957); *Geometrodynamics* (Academic Press, London, 1962); in Relativity, Groups and Topology, eds B. DeWitt and C. DeWitt (Gordon and Breach, New York, 1964).

[21] S. W. Hawking, Nucl. Phys. B144, 349 (1978). S. W. Hawking, D. N. Page and C. N. Pope, Nucl. Phys. B170 [FS1] 283 (1980).

FIGURES

FIG. 1. The dimensionless fluctuation measure $\Delta \equiv \left(\langle\hat{\rho}^2\rangle - \langle\hat{\rho}\rangle^2\right) / \langle\hat{\rho}^2\rangle$ for the Casimir topology, along with $\Delta_{L,\mathrm{Reg}}$. The topology is that of a flat three spatial dimension manifold with one periodic dimension of period $L = 1$. The smearing width σ represents the sampling width of the energy density operator $\hat{\rho}(\sigma)$. Δ is for the complete fluctuations, including divergent and cross terms, while $\Delta_{L,\mathrm{Reg}}$ is just for the finite parts of the mean energy density and fluctuations.

FIG. 2. The finite parts of the mean energy density $\rho^{\mathrm{fin}}(\sigma, L)$ and the fluctuations $\Delta\rho^{\mathrm{fin}}(\sigma, L)$ for the Casimir topology, as a function of the smearing width.